

So, you say that doing math is like playing music? The mathematics classroom as a concert hall

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¿Será posible saber sin ser?
(D'Amore, 2015, p. 153)

Abstract. *In the first part of this article I discuss a dialectical materialist conception of activity. Drawing on this conception of activity, in the second part of the article I introduce a metaphor for mathematics and, in particular, mathematics classroom activity. I suggest conceiving of mathematics classroom activity as a joint endeavor that is carried out together by teachers and students, much like the joint endeavor that is carried out by an orchestra or a musical ensemble that performs, say, a symphony or a string sonata in a music hall or on a subway station. What the orchestra or the musical group produces through its activity is a sensible aural phenomenon: music. In the same way, I submit, mathematics is something sensible, tangible, something that is produced by the joint endeavor of teachers and students and that is simultaneously historical and contemporary, visual and aural, tactile and olfactory, material and ideational, artefactual and symbolic, and kinesthetic and reflective.*

Keywords: mathematics, music, activity, Marx, Hegel.

Sunto. *Nella prima parte di questo articolo prendo in esame una concezione materialista dialettica dell'attività. Basandomi su tale concezione dell'attività, nella seconda parte dell'articolo presento una metafora relativa alla matematica, in particolare relativa all'attività matematica dell'aula. Suggerisco di concepire l'attività matematica in aula come uno sforzo congiunto, che effettuano docenti e studenti, simile a quello di un'orchestra o un insieme musicale che esegue, per esempio, una sinfonia o un concerto di corde in una sala da concerto o in una stazione della metropolitana. Ciò che l'orchestra o il gruppo musicale produce attraverso la sua attività è un fenomeno uditivo sensibile: la musica. Nello stesso modo, a mio avviso, la matematica è qualcosa di sensibile, tangibile, qualcosa che è prodotto dallo sforzo congiunto di docenti e studenti che è allo stesso tempo storico e contemporaneo, visuale e uditivo, tattile e olfattivo, materiale e ideale, artefattuale e simbolico, cinestesico e riflessivo.*

Parole chiave: matematica, musica, attività, Marx, Hegel.

Resumen. *En la primera parte de este artículo examino una concepción materialista dialéctica de la actividad. Basándome en esa concepción de la actividad, en la segunda parte del artículo presento una metáfora en torno a las matemáticas y,*

en particular, a la actividad matemática en el aula. Sugiero concebir la actividad matemática en el aula como un esfuerzo conjunto que llevan a cabo profesores y estudiantes semejante al esfuerzo que hace una orquesta o un conjunto musical que interpreta, por ejemplo, una sinfonía o una sonata de cuerdas en sala de conciertos o en una estación de metro. Lo que la orquesta o el grupo musical produce a través de su actividad es un fenómeno auditivo sensible: la música. De la misma manera, sostengo, las matemáticas son algo sensible, tangible, algo que es producido por el esfuerzo conjunto de maestros y estudiantes y que es simultáneamente histórico y contemporáneo, visual y auditivo, táctil y olfativo, material e ideacional, artefactual y simbólico, kinestésico y reflexivo.

Palabras clave: matemáticas, música, actividad, Marx, Hegel.

1. Introduction

One of the themes that surfaces again and again in the work of Bruno D'Amore is that of “practice.” In his recent work, D'Amore draws on the sociological idea of practice to offer an understanding of mathematics classroom and a typology of practices. He suggests that we consider the mathematics classroom “as a community of shared practices having as its goal the construction of knowledge” (D'Amore & Radford, 2017). He is not alone in this enterprise. A great deal of mathematical educators has followed in the steps of the seminal work of Wegner (1998, 2010).¹

In this article, whose origin is a text written on the occasion of the celebration of Bruno D'Amore's 70th birthday (Radford, 2016a), I would like to reflect on the idea of the mathematics classroom as a community of practices. But I will dare to reverse the elements of the discourse somewhat and argue that what is usually termed “practice” can be better rendered through what Russian dialectical thinkers such as Vygotsky and Leont'ev called *deyatel'nost'* and that is usually translated as *activity*. But, as we shall see in a moment, the problem is not a mere problem of terminology. What lies behind the distinction between practice and activity/*deyatel'nost'* is a radically different conception of humans and what they do. It is, indeed, a profound problem that deals with human reality and human existence.

In the first part of this article I summarize the idea behind *deyatel'nost'*. In the second part of the article I resort to a metaphor to argue that mathematics classroom practice (or mathematics classroom activity) can be considered very much like the musical activity of an orchestra or a musical ensemble in a music hall or on a subway station. First, I would like to start with a culinary anecdote because it was while enjoying a piece of Santa Fe chicken and a glass of wine that I was confronted by the difference between activity and practice.

¹ A recent example is Miranda and Gómez-Blancarte (2018).

2. The Santa Fe chicken

Sometime in the early 2000s Bruno sent me an email to see whether or not one of his PhD students could come to spend some time in my Research Laboratory at Laurentian University. The student, I was told, was working on semiotics. After some email exchanges, we fixed the date for the student's visit to be in the spring. The student was interested in understanding the differences between Raymond Duval's (1995, 1998) approach to signs and human learning, Juan Godino and collaborators' famous onto-semiotic approach (see, e.g., Font, Godino, & Gallardo, 2013), and the Vygotskian semiotic cultural approach that I was trying to articulate (Radford, 2006; for recent formulations, see Radford, 2008, 2014a, 2018a, 2018b).

After some weeks of intense discussions around mathematics classroom videos and the analysis of students' productions, we made some progress. The difference between the approaches started to emerge with more clarity. But, of course, the student had to return to his country. So, to give us an opportunity to summarize our mutual learning, on the eve of his departure, I made a reservation at a restaurant not far from the university. In the tremendously slow melting of the snow and a recurrent cold wind that contrasted with the regular lengthening of the days, we could predict the end of the winter and the advent of the spring.

That evening, in a quiet corner of the restaurant, with his kind smile and manners, working on a robust and colorful salad, Giorgio Santi was sitting in front of me. With a glass of wine in his hand, he mentioned that differences between the onto-semiotic approach and the cultural one cannot be found in the concept of activity. Although the onto-semiotic approach may not have "activity" among its main theoretical constructs, it resorts systematically to the concept of "practice." "In the onto-semiotic approach, they talk about the mathematician's practice," Giorgio contended. Giorgio was right (Santi, 2011).

I remember that I tried to counter by saying that these two concepts were not the same thing. But I had to surrender. I could not explain why. The differences were not clear. What, indeed, do we mean by practice and by activity? The discussion ended without conclusion. It ended with a sense that, to make the differences visible, I had to keep thinking about the meaning of activity. To argue, as I did, that I was using activity in Leont'ev's (1978) sense was not enough.

3. *Deyatel'nost'*

Activity is one of those catch words that is used in the human sciences (e.g., in philosophy, psychology, and education) in descriptions and investigations of human phenomena. All that we humans do is, indeed, immersed in an activity of some sort. But what is the status of this activity? What exactly does activity

mean?

Activity has a definite and precise meaning in new dialectical materialism. Such a meaning does not only allow us to overcome the dualism between the individual and the social but allows us also (and overall) to conceive of human action as enmeshed in a conceptuality that is essentially cultural, social, historical, and political. One of the greatest advantages of such a perspective on activity—especially in current times where individualism has become the trademark of everyday life—is that it makes room to see human undertakings (like doing mathematics in a classroom) as an essentially cultural and historical collective endeavor. Let me dwell on the dialectical materialist meaning of activity. I start first with some comments on the non-dialectical materialist meaning of activity.

The Merriam-Webster online dictionary gives the following definition of activity: “Something that is done as work or for a particular purpose” (<http://www.merriam-webster.com/dictionary/activity>).

This definition highlights two things. First, activity is about doing. Second, it is about doing something purposely. It is from this sense of activity that the usual conceptions of activity derive. For instance, activity is conceived of as a series of actions that an individual performs in the attainment of his or her goal. This is how activity appears in Piaget’s work. For instance, in *La formation du symbole chez l’enfant*, Piaget says: “Quand l’assimilation l’emporte sur l’accommodation, l’activité *du sujet* s’oriente par cela même dans le sens du jeu” (Piaget, 1978, p. 12; my emphasis). This concept of activity can even go a step further and include the actions of *various* “actors” or “agents.” If we continue walking along this path, we end up here with the concept of the activity of a *collective*. Here, activity usually comes to mean a trans-subjective coordination of actions and *reactions*.

The German and Russian languages have a specific term for the Piagetian and collectivist conceptions of activity as being simply busy with something: *Aktivität*, and *aktivnost’*, respectively (Roth & Radford 2011).

There is, however, another sense for activity, which is the one emphasized in dialectical materialism, where activity does not merely mean “to do, and be busy with, something.” The German and Russian languages have a specific term that better conveys the idea of activity of dialectical materialism: *Tätigkeit* (in German) and *deyatel’nost’* (in Russian), one that puts at its heart the idea of *life*—life not in a biological sense, but in a cultural-historical one. English and other languages do not have a specific term for this sense of activity and use “activity” for *aktivnost’* and *deyatel’nost’* and for *Aktivität* and *Tätigkeit*. In doing so, the very important meaning conveyed by the term *deyatel’nost’* or *Tätigkei* is buried.

When Leont’ev tries to define activity (*deyatel’nost’*), he resorts to the idea of activity as a *unit of life*. Leont’ev says that activity

is a molar, not an additive unit of the life of the physical, material subject. (...)

activity is not a reaction and not a totality of reactions but a system that has structure, its own internal transitions and transformations, its own development. (1978, p. 50)

Let me insist: Activity as *Tätigkeit* or *deyatel'nost'* is a *form of life*. More precisely, it is a social form of joint endeavor that comprises self-expression, intellectual and social development, and aesthetic enjoyment.

In this line of thought, activity refers to what Aristotle describes in his *Metaphysics* as a process—an unfolding energy—something that in modern terms we may call a *dynamic system* that, instead of being simply object- or goal-oriented, is geared to the satisfaction of collective needs and the self-expression of the individuals.

When I was finishing my Santa Fe chicken that memorable evening with Santi, I had not realized yet that behind this dialectical idea of activity rests a specific conception of the human. I had not realized yet that the dialectical idea of activity can only be understood if we think of it along with the corresponding anthropological conception of the human. It took me years to realize it. It is perhaps the French philosopher Frank Fischbach who helped me the most to understand it during the lengthy conversations that he had when he was visiting Laurentian University and my research lab. The interested reader can watch Fischbach's brilliant lecture on subjectivity in our site: <http://penseeetculture.ca/2015-16-conferences/>

The deep relationship between activity and the concomitant concept of the human was something that Marx started articulating in the *Parisian manuscripts*, i.e., the *1844 economic and philosophical manuscripts* (Marx, 1988) and that he continued developing in the sketchy notes he wrote down in *Theses on Feuerbach*, and in the famous book *The German Ideology* (Marx, 1998). Marx's starting point was the effort he deployed to go beyond a naive and concrete view of reality and the individual as conveyed by Feuerbach's materialism. In the first of the *Theses on Feuerbach*, Marx argues that

The chief defect of all previous materialism—that of Feuerbach included—is that things [*Gegenstand*], reality, sensuousness are conceived only in the form of the *object*, or of *contemplation*, but not as *human sensuous activity* [*Tätigkeit*], not [as] *practice*, not [as] subjectively. (Marx, 1998, p. 572; emphasis in the original)²

According to Marx, the “contemplative” stand of Feuerbach's approach led him to considered reality as something *merely given*, as standing in a neutral magnificence in front of the individual. In the *German Ideology*, Marx

² The excerpt reads as follows: “Der Hauptmangel alles bisherigen Materialismus (den Feuerbachschen mit eingerechnet) ist, daß der Gegenstand, die Wirklichkeit, Sinnlichkeit, nur unter der Form des *Objekts* oder der *Anschauung* gefaßt wird; nicht aber als *sinnlich menschliche Tätigkeit, Praxis*; nicht subjektiv.” (http://www.mlwerke.de/me/me03/me03_005.htm)

elaborates the idea and contends that

He [Feuerbach] does not see that the sensuous world around him is not a thing given direct from all eternity, remaining ever the same, but the product of industry and of the state of society; and, indeed, [a product] in the sense that it is an historical product, the result of the activity of a whole succession of generations, each standing on the shoulders of the preceding one, developing its industry and its intercourse, and modifying its social system according to the changed needs. Even the objects of the simplest “sensuous certainty” are only given him through social development, industry and commercial intercourse. The cherry-tree, like almost all fruit-trees, was, as is well known, only a few centuries ago transplanted by *commerce* into our zone, and therefore only *by* this action of a definite society in a definite age has it become “sensuous certainty” for Feuerbach. (Marx, 1998, p. 45; emphasis in the original)

Reality and its objects, as Marx says in the first thesis, need to be understood in terms of “*human sensuous activity*,” within the scope and extent of human activity. *And vice-versa*: Humans need to be understood in terms of their activities, for, in Marx’s conception, through activity the individuals are not only doing something, producing something; through activity, at the same time, the individuals act on themselves, and by acting on themselves, they “transform” themselves (Fischbach, 2015, p 29). Thus, all human capacities (intellectual, sensorial, imaginative, etc.) are not *given entities*, but the *product of cultural-historical activity*. For, according to Marx’s (1988, 1998) Spinozist stance, humans are considered to be part of nature: They are considered *natural beings*, that is they are sensible beings. And to say that humans are sensible beings means that humans are unavoidably *affected* by the other parts of nature: by things and people, by what we and other people do.

In Marx’s account, the most fundamental human trait is the capacity for self-activation. Humans are beings of action. They have to activate themselves to fulfil their needs, for, as other natural species (like chimpanzees), humans are beings of needs. And the satisfaction of needs is found in objects outside of themselves. This vital activation that allows humans to meet their needs (needs of survival and also artistic, spiritual, intellectual, and other needs created by/in society), leads them to engage actively in the world. Through this activity they *produce*. But what Marx tries to say again and again in the *German Ideology* is that what humans produce to fulfil their needs occurs in a social process that is, at the same time, the process of their production *as individuals* and the production of *their own existence*. This production is a process that

must not be considered simply as being the reproduction of the physical existence of the individuals. Rather it is a definite form of activity of these individuals, a definite form of expressing their life, a definite *mode of life* on their part. As individuals express their life, so they are. What they are, therefore, coincides with their production, both with *what* they produce and with *how* they produce. (Marx, 1998, p. 37)

In dialectical materialism, the name of this process is *deyatel'nost'*, that is, *activity*. This is why, from this perspective, sensuous, material activity is considered the ultimate field of aesthetic experience and cognition, and that discourse-oriented, and practice-oriented, and *deyatel'nost'*-oriented ways of theorizing are not the same (Radford, 2016b).

The general use of the concept of activity in other areas of scholarly research where activity appears as *aktivnost'* or *Aktivität* (i.e., as a sequence of actions) misses the specific sense it has in dialectical materialism (or the materialism of practice, to use Marx's term) where reality and individuals can only be understood in terms of their activities, where individuals create activity and, in turn, activity creates them.

One important consequence of this theoretical stance is that the individual's existence cannot be conceived of as a substantial entity, produced from within, as articulated by the humanist trend of the Enlightenment. In the Enlightened humanist trend, the individual is the constitutive locus of knowledge and affect. What Marx asserts, by contrast, is that the individual's existence is *relational* through and through. It appears to be profoundly linked to an ensemble of relationships with other parts of nature—including social relationships—and is based on culturally and historically constituted conditions of life. This is what Marx says in the famous sixth thesis on Feuerbach, when he defines that which makes us human: “But the essence of man [sic] is no abstraction inherent in each single individual. In its reality it is the ensemble of the social relations” (1998, p. 570). From this idea of humans, Marx can affirm that “Essence, therefore, can be regarded only as ‘species’, as an inner, mute, general character which unites the many individuals in a *natural way*” (p. 570; emphasis in the original). In other words, the social relations that constitute our “essence” (Marx was using the idealist terminology to turn it upside down) do not merely sprout spontaneously in our contact with others. Those social relations—relations of friendship, of love, of an ethical nature, etc., are historical and cultural (Radford, 2015; Ratner, 2000). It is against this cultural-historical background that we come to enact the social relations, always in novel and idiosyncratic manners.

In *A cultural historical perspective on teaching and learning* (Roth & Radford, 2011) Michael Roth and I tried to look at classroom activity through the lens of *deyatel'nost'*. To do so, we resorted to Leont'ev's (1978) seminal work. In articulating a psychological approach based on the idea of activity as *deyatel'nost'*, Leont'ev (1978) highlighted some of activity's basic components: An activity for him is characterized by its object and its motive. The object and motive of an activity are the engines that keep activity in motion. In practice, in the pursuit of the activity's object, individuals break down the object into a sequence of goals with which actions are associated. He referred to the material conditions through which the actions occur as *operations*. In the Supplement to his important 1978 book—a supplement

dedicated to educational matters—Leont’ev discusses the conditions under which a certain theoretical learning content can be meaningfully perceived or attended to by the student. He contends that

in order that the perceived content be recognized, it is necessary that it occupies the structural place of a direct goal of action in the subject’s activity, and thus that it appears in a corresponding relation to the motive of this activity. (Leont’ev, 1978, p. 153)

It is hence through activity and the structural interconnection between motive, object, goals, and actions that the learning content becomes disclosed to the student’s consciousness.

Activity Theory, as this approach has come to be known, has had an important impact on education in general and mathematics education in particular. Yet, in focusing on the *procedural* aspect of activity, activity is reduced to its operational and functional dimension, eradicating the aesthetic and political dimensions of action and creation, culminating unfortunately in a dull technological account of what was originally thought of as the sensible experience of *life*—human life.

How, then, could we recover the idea of activity in the sense of *deyatel’nost’*? It is here where I need to turn to the metaphor of music.

4. Mathematics as a sensible, material phenomenon

A few years after Santi’s visit to my laboratory, as part of the research activities of the Kaput Center, Stephen Hegedus organized a Colloquium on Symbolic Cognition. To avoid the mundane distractions of busy cities, Hegedus organized the colloquium in a secluded and remote cottage-hotel in Vermont.

Like many other participants of the Colloquium, I landed at the Boston Airport. A taxi was waiting for me. As I got into the taxi, the driver let me know that we had to wait for another participant who was supposed to arrive anytime. “And who is it?” I asked. The driver consulted his log sheet and responded, “Mr. Colette Laborde.” I corrected him and said, “Mrs. Colette Laborde.” He checked his log sheet again and said emphatically, “Mr. Colette Laborde,” and added “and he is travelling with skis.”

While we circled the various doors of Boston Airport looking for a man with skis, I tried to convince the driver that there was no one called Mr. Colette Laborde attending the meeting; I drew on Toulmin’s model of argumentation, backed up my claims with carefully chosen warrants, but failed miserably. Then I spotted Colette in front of one of the doors. I waved and she waved back. The taxi stopped and she got in. She was not bringing skis. The driver was surprised, called headquarters, and complained. Then we left. We drove for about 45 minutes and arrived at night at the cottage-hotel. Most of

the participants had already arrived. The group was composed of about 15 researchers—including Norma Presmeg, Adalira Sáenz-Ludlow, Nathalie Sinclair, Hyman Bass, David Tall, Gerald Goldin, and Gari Davis.

As part of the program, in the afternoon of one of the first days of the Colloquium, we had a special guest: neuroanthropologist Terrence Deacon, author of a very interesting book: *The Symbolic Species* (Deacon, 1997). Deacon got out of a car, went directly to the main room, unpacked some notes, took two long breaths, and gave a remarkable talk about chimpanzees' language learning.

The following day I was having breakfast with some mathematicians and mathematics educators at a table by the window. We could see the beautiful, totally white, landscape. It had been snowing without interruption for days. It was January and it was cold. I do not remember what brought us to discuss the nature of mathematics. Maybe it was a good night's rest, or the interesting discussion on symbols the day before after Deacon's talk, or both. I ventured to mention that mathematics could not be equated to symbols on the pages of a book. The symbols on the pages of a book are exactly that—symbols, or marks, to put it more bluntly.

To support my claim, I resorted to music. In the same way that there is no music in a score sheet, there is no mathematics in the pages of a mathematics book. Music is what we *hear* when people play instruments. Music is a sensuous aural entity.

Mathematics is also a sensuous entity: It *appears* much in the same way as music appears when an orchestra plays, say, a symphony. Like music, mathematics is something that *appears* as students and teachers engage in classroom activity. What appears in the mathematics classroom is not exactly an aural phenomenon or a visual or tactile or olfactory one. Yet, something appears (and perhaps is something that is all of that: visual, tactile, olfactory, aural, material, artefactual, gestural, and kinesthetic) and, being all of that, becomes an object of consciousness and thought. In this materialist and phenomenological line of thought, school mathematics is what is made sensible through the teachers-and-students' activity—and mathematics is what is made sensible through the mathematicians' activity.

In 2006, I had the opportunity to participate in a workshop on the history of mathematics and mathematics education that Fulvia Furinghetti, Hans Niels Jahnke, and Jan van Maanen (2006) organized at the *Mathematisches Forschungsinstitut Oberwolfach*, situated in the German Black Forest. The Oberwolfach Institute offers a space where several workshops can be conducted simultaneously. In the evening, participants gather together after supper in the various ample living rooms of the facility to get some drinks and keep talking about mathematics. On one of those evenings, two mathematicians were discussing a mathematical problem in a very animated manner. I found it interesting that, in contrast to other mathematicians, these

two mathematicians were not writing. At a certain point, one of them got up and went to a fridge to grab two beers. I took the opportunity to turn to them and ask about the problem in which they were immersed. I mentioned that I found it peculiar that they were not writing. “Oh!” one of them responded, “we are tired of writing.” These mathematicians were doing mathematics by coordinating gestures, talk, and perception. If the two mathematicians had remained silent, sitting in front of each other while doing nothing, mathematics would have not appeared.³

To continue with the Vermont metaphor, we need to make some distinctions. I am not saying that mathematics and the activity that produces it are the same, much as the music that is being heard cannot be confounded with the orchestra’s activity. Yet, both are deeply intertwined: the activity and what the activity produces. We cannot extract one from the other: We cannot extract the mathematics from the classroom activity, as we cannot extract the orchestra’s activity from what we hear. They are glued together in a fundamental sense. As the activity unfolds, mathematics appears—much as, for example, a symphony appears as the orchestra’s activity unfolds. Unfolding and appearing have to be understood here in a *dialectical* relationship. The unfolding affects, moves, and transforms the appearing, and the appearing affects, moves, and transforms the unfolding.

But things do not merely happen or appear out of the blue. The sound that is produced by a violin, for example, has its source in the instrument. It is in the instrument but in a *potential* manner only. The sound may or may not be produced. And *if* it is produced, it can be produced in countless ways. It is both contingent and historically bound. In being produced, the sound materializes or *actualizes* that which was potentiality or pure possibility. (I am drawing here on Aristotle’s [1998] *Metaphysics* and Hegel’s [1991] *Encyclopaedia* to distinguish between potentiality and actuality; for details see Radford, 2013). In this line of thought, we can consider both mathematics and music as sensuous evolved forms of something that before being materialized and coming into sensible existence, was *general*. The general is *formless*. It belongs to the realm of potentiality. Yet, it is not a Platonic Form. The realm of the potential belongs to an always changing immaterial sphere of culture that is *intertwined* with the material world of objects and human actions. This immaterial sphere of culture is part of what Marx (1998, p. 75) called the “inorganic” realm of nature and it is also part of the conditions out of which human existence is crafted. This sphere cannot be sensed by we humans through our culturally and historically evolved senses and sensations. Can we

³ We can go one step further and ask: Can mathematics appear by just *thinking* about it, like in going through the steps of the resolution of a problem or thinking about the coefficients of a transition matrix in a Markov process? Yes. Thinking is already a form of activity (Wertsch, 1991, 1998), or as Sfard (2008) puts it, an individualized form of dialogical or communicative activity.

sense or perceive or touch the Pythagorean theorem *as such*? We cannot. Can we hear Beethoven’s 7th symphony *as such*? We cannot. To become the object of consciousness, feeling, and thought, the general (mathematics *as general*; music *as general*) has to be set into motion to transform it into something sensible, and appear. Its appearance is what in Hegel’s terminology is called a *singular*. The singular is the appearance of the general through the mediation of human activity.

To make the previous idea clearer, let me turn to Beethoven’s music—which Adorno (1998, p. 224) considered “bourgeois music at its very height.” Let me turn to Beethoven’s 7th symphony. As we know, Beethoven’s 7th symphony has four movements: Poco sostenuto–Vivace, Allegretto, Scherzo, and Allegro. Table 1 presents the duration of the symphony as conducted by two orchestra directors who are considered to be among the best 20th century Beethoven specialists: Herbert von Karajan and Leonard Bernstein.

Table 1
Total time of two famous recordings of Beethoven’s 7th symphony

	Herbert von Karajan (Berlin Philharmonic Orchestra, 1963 recording)	Leonard Bernstein (New York Philharmonic Orchestra, 1958 recording)
Poco sostenuto–Vivace	11:25	12:27
Allegretto	8:02	9:44
Scherzo	7:50	8:23
Allegro	6:37	7:27
Total time	33 min 54 s	38 min 01 s

Bernstein’s recording is 12% longer than Karajan’s. And I think that the reader would agree with me that 12% is a lot. It is not an insignificant difference. Which one is the true 7th symphony? Neither of them. The 7th symphony as such is a *general*. Bernstein’s and Karajan’s recordings are *materializations* of this general; that is, they are singulars, or, in other words, they are appearances of the general. In musical jargon, both are interpretations—interpretations of a general archetype: the 7th symphony *as such*.

An objection could be presented to my argument. It could be said, indeed, that the *exact* 7th symphony is what we would hear, if Beethoven conducts it. This is a sound objection. Let us imagine that Beethoven conducts it two consecutive days. Would the 7th symphony sound exactly the *same*? In all likelihood, no. Let us take a simpler example. In April 1902 Sergei Rachmaninov composed a song called “Lilacs” inspired by a Russian poem.

He recorded two transcriptions for piano, one on December 27, 1923, and the other on February 6, 1942. Despite being a short musical piece, both recordings by the *same composer* are not the same (see https://youtu.be/aa2k_s2ZxN8 for the 1923 recording and <https://youtu.be/72xh91KTOOA> for the 1942 recording). Which is the true one? As is in the case of the 7th symphony, neither of them.

In a TV program, *The Creative Performer*, Bernstein (1960) contends that the musical score may contain as many instructions as the composer wishes—e.g., about tempo or emotions (like in *Allegretto*)—yet be rendered very differently by different conductors. Talking about two interpretations of Beethoven’s “The Eroica,” he asks: “Now how can those two performances be so different?” He goes on to say that

After all, we’re dealing with a score of published records of certain utterances by Beethoven. Clearly and unmistakably conveyed by the hieroglyphic of musical notation, shouldn’t it then mean the same thing to all men? Yes, but remember that this score, only half exists, it’s only a score, a printed record. Until performers take it, assimilate it, combine their energies with its dormant ones and spark it to life by performing it. (Bernstein, 1960)

As a semiotic text the score refers to a general that is only “half alive.” It is the *activity* of the orchestra that sparks the score to life. In coming into life, a human factor unavoidably appears. But instead of being something that we need to reduce or discard, it is a factor that is part and parcel of the general and its materialization. Bernstein continues:

The explanation lies in the human factor. A factor that cannot possibly remain glued to any mathematical formula throughout a piece of music. At best, that metronome mark of 60 [in the first notes of *The Eroica*] can mean only in the neighborhood of 60. After all, there’s our friend “Conductor Z” [a hypothetical lyrical conductor that Bernstein opposes to a hypothetical dramatic “Conductor A”], who understands the tempo of the main theme to lie just a shade under 60, so that the theme will emerge more gently, perhaps nobler, more imposing [than in Maestro A’s interpretation]. (Bernstein, 1960)

As a general, a symphony is a possibility, and not something that is there to be matched. As all generals, “Lilacs” and the 7th symphony are general archetypes, generative capacities. They are possibilities that allow musicians to materialize music. In materializing them, “Lilacs” and the 7th symphony can be heard, and then modified. Musicians may experiment with them and bring music to new forms, thereby creating new archetypes, new generals. And the same is true of mathematics. Mathematics is not something to match or to repeat, but a generative capacity to do and reflect on things in a certain way. The hundreds of proofs that we know of the Pythagorean Theorem followed a similar fate. The materialization of a proof served as an experimental starting point to try to find other proofs. In music, in mathematics, in law, etc., there is a dialectical relationship between a general and its materializations. The

general makes the materializations possible, but then the materializations can be pushed beyond what is known and enter into new creative domains.

At any rate, “Lilacs” and the 7th symphony are inscribed within a certain cultural tradition. For instance, the 7th symphony belongs to a musical tradition of leisurely symphonic prologues, a romantic paradigm, an increasing focus on rhythm and the smart use of available musical artefacts (e.g., metronomes for measuring tempo), among others (Marx & Burnham, 1997; Will, 2004). Rather than existing in itself and by itself, the general (in music or mathematics) is to be found in culture and history.

I would like to summarize these ideas by saying that the singular is the appearing of the general (i.e., knowledge: mathematical knowledge, musical knowledge, etc.).⁴ In other words, the singular is the coming into existence of the general as an evolved ontological form transformed under the force of an activity (in the sense of *Tätigkeit / deyatel'nost'*, as defined above). That this activity (the *particular* in Hegel's terminology) is not merely an *Aktivität* or *aktivnost'* is shown by the fact that the *Aktivität* or *aktivnost'* would produce something else, something like the 7th symphony interpreted by programmed artefacts and mechanical devices only. Such appearing would in fact lack exactly that which makes *Tätigkeit / deyatel'nost'* what it is, namely *human, natural life*.

5. The classroom as a concert hall

My incursion into music in the previous section was motivated by the question of activity in teaching-learning mathematics. In Section 3, I made a distinction between *Aktivität* or *aktivnost'* on the one hand, and *Tätigkeit* or *deyatel'nost'* on the other. I argued that classroom activity in the sense of *Tätigkeit* or *deyatel'nost'* would be conducive to a vibrant mathematics classroom and a model for the kind of activity that could be encouraged in schools. To better understand the characteristics of this activity, I suggested that music could be an interesting example to consider. Indeed, when orchestras, musical ensembles, etc., successfully play a certain piece of music, the musicians engage productively in a profound, collective activity, in *Tätigkeit* or *deyatel'nost'*. In this case, musicians listen and react to each other and form a tuned whole where an energy glues them together and the music that they produce simply flows. And this is pretty much what happens in the best mathematics classrooms that I have seen in schools. In these cases, teachers and students do mathematics the same way that musicians play music. In these cases, the mathematics classroom works as a concert hall.

⁴ In a 1943 letter to Rudolph Kolisch about Beethoven's music, while he was in exile in Los Angeles, Adorno writes: “And like you, I believe in the strict knowability of music—because music is itself knowledge, and in its way very strict knowledge” (Adorno, 1998, p. 180).

To spell out the similarity between playing music and doing mathematics in school, in Section 4 I outlined an ontology in which mathematics and music are considered as something *general* in Hegel's sense; that is, as culturally and historically constituted generative capacities to engage in the world in certain ways. Phenomenologically speaking, through human activity, these generative capacities become *materialized*. They become singular evolved forms of something general that before being set into motion by activity were potentiality, pure possibility. It is through activity that mathematics and music come into sensible existence as something that can be now an object of feeling, thought, consciousness, critique, and transformation. The outlined ontology rests hence on three entities in dialectical relationship: general–particular–singular or knowledge–activity–knowledge materialization. That what makes the similarity between mathematics and music possible is this common ontological structure.

In this section I would like to discuss in more detail the *nature* of the activity in the aforementioned ontological structure, focusing in particular on classroom activity. Classroom activity can take different forms, offering the students different kinds of mathematical experiences. Underlying each kind of classroom activity and the ensuing mathematical experience is a pedagogical stance. One of them, the most popular one, at least in certain milieus, is direct teaching. Here the teacher simply tells the students what to do and the students obediently do as the teacher says. This activity looks pretty much like the mechanical one of the example that I mentioned in the previous section, where people do not really connect or work *together*. They simply do things and follow rules, as in *Aktivität* or *aktivnost*'. It is a poor activity, much as the activity of an orchestra playing a musical piece where the musicians do not connect in a human way—something terribly boring and unexciting. It is *lifeless activity*, bad activity.

What could be a good mathematical activity? I think that a first general answer is an activity in which students *engage meaningfully in the mathematics*. Engaging meaningfully entails the definite presence of an active side of the students in mathematical inquiries and problem posing and solving. Engaging meaningfully entails the creation of space for debates, and for exploring conjectures, theorems, etc. I think that there would not be much disagreement about this. By contrast, engaging meaningfully *in the mathematics* is a much trickier idea. Indeed, whose mathematics are we talking about here? Is it the students' mathematics, that is, the mathematics that they produce on their own? Constructivism is the educational paradigm that answered this question in the affirmative (see, e.g., Glasersfeld, 1995), and child-centered pedagogy is the transposition and operationalization of this answer into pedagogical action. In Radford (2014a) I suggested that child-centered pedagogy ends up in alienation. The child is alienated from culture, history, and society. In the child-centered pedagogy, the child remains

prisoner of the confines of her own subjectivity, disconnected from the cultural world, living a sad and monotonous life in a solipsistic shell.

From this discussion two main courses of pedagogical actions emerge:

- (1) one that emphasizes the subjective side of knowing and ignores the already historically present structuring possibilities of knowledge (as conveyed by child-centered pedagogy), and
- (2) one that, emphasizing the structuring possibilities of knowledge, considers mathematics as an ensemble of procedures to follow and ignores the subjective side of knowing (as in direct teaching pedagogy).

The first choice revolves around the pole of *subjectivism*. Mathematics is what the students produce. The second choice revolves around the pole of *reification*, where mathematics appears as something independent of the students.

Naturally, there are plenty of pedagogical positions between subjectivism and reification. Most of them start from the students' mathematics hoping, as Piaget did, that they will end up naturally doing the mathematicians' mathematics.

A few years ago, Neyland (2003, 2004) suggested something different—something that combines the subjective side of doing mathematics without however, and this is the novelty, ignoring the already historically present structuring possibilities of knowledge. To do so, he turned to the metaphor of music—although not any music. He suggested jazz: jazz as an exemplar of what mathematical activity—good classroom mathematical activity—could look like. To understand his choice, we have to bear in mind that in Neyland's account, playing jazz is done by combining two elements:

- (1) a structural element that is already there, in musical culture, structuring music before the jazz players start playing, such as “harmonic” and “modal” structures (the reification pole in the terminology introduced above); and
- (2) improvisation (the subjective pole). Improvisation may happen in what Neyland calls “playing outside” (2003, p. 6); that is, playing *outside the structure*.

In Neyland's eyes, jazz offers the musicians the possibility to go beyond structure and shine subjectively. Indeed, playing outside the structure allows musicians to emancipate themselves from the underlying structures, finally moving freely of constraints. “It is using the structure to do exactly what it was set up not to do. (...) It leads to the structure being challenged” (Neyland, 2003, p. 6).

So, Neyland turned to music to offer us an example of a *music genre* that may help us overcome the antinomies that arise from the subjectivism–reification opposition that seems to permeate, more often implicitly than explicitly, our pedagogical choices. Jazz is certainly the music genre that

draws minimally in the structure, while making at the same time room for the subjective expressivity of the player. “Jazz combos,” Neyland (2003) says, “aim to achieve an optimally minimal structure that supports a maximal degree of creativity” (p. 3).

Although I certainly find interesting the metaphor of jazz as an exemplar of good classroom activity, I do not think that other genres are less interesting. Perhaps the most important aspect of Neyland’s jazz metaphor (Neyland, 2003, 2004) rests on the dramatic way in which it brings to the fore the antinomies of the subjectivism–reification opposition, where jazz lets us dream that we can escape structure—at least to a certain extent. However, the emancipatory dream that jazz can offer is just that, a dream, for “If she [the jazz player] plays outside [the structure] too much, the [jazz] combo ceases to function and collapses in harmonic disarray” (Neyland, 2003, p. 6). In other words, you can live out of the structure ephemerally only. Jazz embodies the emancipatory fantasy that philosophers of the 18th century Enlightenment dreamt of.

Now, if it is not jazz, then what is it? I do not think that it is a question of music genre. I think that the question is about how music is played and mathematics is done. The question, I would like to submit, is the manner in which people *relate* to each other, and how they position themselves and are simultaneously positioned by others while playing music or doing mathematics in the classroom. The question, to say it in other words, is about the *forms of human collaboration* and the *forms of* (musical, mathematical) *knowledge production* (Radford, 2014a) around which a *common work* is sensibly produced, may this be a symphony, a piece of jazz, a string concerto, the mathematical investigation of a problem, etc. It is here, in the intellectual-corporeal-affective production of a *common work*, through a collective endeavor where students and teachers work and learn together and critically agree and disagree, where teachers and students show responsibility, solidarity, and care for each other, that we may be able to recognize the chief characteristics of *Tätigkeit or deyatel’nost’*, and that doing mathematics may be similar to playing music and that the mathematics classroom could resemble a concert hall.

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