Algebra-related tasks: Teachers’ guidance in curriculum materials

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Abstract. *Researchers and curriculum frameworks recommend engaging students with algebra-related topics starting from primary school. Even though algebra-related topics hold a place in curriculum materials, little research has focused on the nature of the opportunities in textbooks and on the guidance for teachers. Curriculum materials present the learning opportunities for students but also indicate to teachers what mathematics to teach and how to enact the intended opportunities. This paper presents an analytic framework for examining the guidance offered to teachers to enact the algebra-related tasks in curriculum materials. It presents findings from the application of the framework to examine the teachers' guides of the textbook series used in the state schools in Cyprus. Implications for curriculum design, research, and practice are discussed.*

Keywords: algebra-related topics, curriculum materials, teachers’ guidance, curriculum design.

Sunto. *I ricercatori e le indicazioni curriculari raccomandano di coinvolgere gli studenti su temi legati all'algebra fin dalla scuola primaria. Anche se i temi legati all'algebra hanno un posto nei materiali curriculari, poca ricerca si è concentrata sulla natura delle opportunità offerte dai libri di testo e sulle guide offerte agli insegnanti. I materiali curriculari presentano alcune opportunità di apprendimento per gli studenti, ma indicano anche agli insegnanti quale matematica insegnare e come attuare le opportunità previste. Questo articolo presenta un quadro analitico per esaminare le guide offerte agli insegnanti per promuovere attività legate all'algebra nei materiali curriculari. Esso presenta i risultati dell'applicazione di tale quadro per esaminare le guide degli insegnanti dei libri di testo utilizzati nelle scuole statali di Cipro. Vengono inoltre discusse le implicazioni per la progettazione curricolare, la ricerca e la pratica.*

Parole chiave: temi legati all’algebra, materiali curriculari, guide per gli insegnanti, progettazione curricolare.

Resumen. *Los investigadores y los marcos curriculares recomiendan comprometer a los estudiantes con temas relacionados con el álgebra a partir de la escuela primaria. Aunque los temas relacionados con el álgebra ocupan un lugar en los materiales del plan de estudios, poca investigación se ha centrado en la naturaleza de las oportunidades de los libros de texto y en la orientación para los profesores. Los materiales curriculares presentan unas oportunidades de aprendizaje para los estudiantes, pero también indican a los profesores qué matemática enseñar y cómo* *actuar en la promulgación de las oportunidades previstas. El presente artículo ofrece un enfoque analítico para examinar la orientación ofrecida a los profesores para realizar las tareas relacionadas con el álgebra en los materiales del currículo. Presenta los resultados de la aplicación del enfoque para examinar las guías de los profesores de una serie de libros de texto utilizados en las escuelas estatales en Chipre. Se discuten también las implicaciones para el diseño curricular, la investigación y la práctica.*

Palabras clave: temas relacionados con álgebra, materiales del plan de estudios, orientación de los maestros, diseño curricular.

1. Introduction

Algebra and algebraic thinking are essential in engaging and understanding fundamental concepts both in mathematics and in other scientific domains (Usiskin, 1995). Algebra as a topic in school mathematics has been traditionally associated with the upper school levels. However, both researchers and curriculum frameworks recommend that students should be offered learning opportunities that can prepare them for formal algebra learning from primary school years (e.g. NCTM, 2000; Stacey, Chick, & Kendal, 2004).

Students build critical foundations for algebra when they learn arithmetic with understanding, by representing and justifying general relations between numbers and properties of operations (Carpenter, Franke, & Levi, 2003). In this way, they also start to understand the nature and importance of proof, and engage in sense-making activities (Carpenter, Levi, Berman, & Pligge, 2005). In addition, the nature of school algebra broadens, by providing coherence and depth in the mathematics curriculum, and prevents students' alienation resulting from a late and abrupt transition to algebra in secondary school.

The relevant field of research has so far been concerned with intervention studies that examined primary school students' capacity to engage with algebra-related topics. The findings have shown that even young students engaged successfully with algebra-related topics in supportive classroom environments (e.g. Blanton & Kaput, 2004; Carpenter et al., 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006). Specifically, students were able to identify relations, reason about quantities, make justifications and generalizations using different representations, and work with patterns, functions and story problems (Dougherty, 2008; Moss & McNab, 2011).

Beyond the research studies, curriculum documents in various countries also provide opportunities for primary school students to engage with algebra-related topics (Cai, Lew, Morris, Moyer, Ng, & Schmittau, 2005). Apart from presenting the potential learning opportunities, the curriculum materials can also serve to educate teachers and improve classroom instruction (Ball & Cohen, 1996; Davis & Krajcik, 2005). Teachers' decisions about what mathematics to teach, when and how to teach it are usually mediated by the curricular materials they use (Haggarty & Pepin, 2002; Porter, 2002). Research on curriculum materials can also offer reliable feedback to curriculum developers (Cai & Cirillo, 2014).

So far, little research has been done on the nature of the algebra-related opportunities in curriculum materials and the guidance for teachers to enact these opportunities in the classroom. There is evidence to suggest that primary school teachers tend to recognise algebra-related tasks by the existence of letter symbolism or symbol manipulation (Stephens, 2008). Yet, this conception of algebra does not reflect the breadth of algebra-related topics currently mentioned in the literature. This raises concerns about how teachers engage students with tasks that have the learning potential to prepare them for algebra. Hence, there is even more need to explore what kind of guidance is provided for teachers to enact these tasks in the classroom.

The field of mathematics education lacks an analytic framework that could help in systematically examining the guidance for enacting the algebra-related opportunities in classroom. It is particularly worth conducting such investigations on topics in relation to which students and teachers face significant difficulties (Stylianides, 2014). This paper presents an analytic framework for examining the guidance for teachers and contributes in developing understanding and insight into the guidance that could be meaningful and educative for teachers to implement the tasks in classroom. The analytic framework aimed to identify the algebra-related tasks in textbooks and afterwards, to examine the respective guidance in teachers' guides.

2. Analytic framework

Below, we present the conceptualization of algebra-related tasks and components of guidance in curriculum materials.

2.1. Algebra-related tasks

Algebra-related tasks are defined, in the context of this study, as tasks that provide opportunities for students' engagement with algebraic ideas relevant to primary mathematics. Letter symbolism solely did not stand as a criterion for identifying algebra-related tasks since alphanumeric symbolism is only one of the semiotic forms of algebra (Radford, 2010).

According to Demosthenous and Stylianides (2014, 2017), algebra-related tasks are grouped into the following three categories according to the relations between numbers and quantities in the tasks: arithmetically-situated relations, rule-based relations and known-unknown relations. Arithmetically-situated relations tasks correspond to what is referred to in the literature as generalized arithmetic (Carpenter et al., 2003; Kaput, 2008). These tasks focus on the structure of arithmetic by attending to the behavior of arithmetic operations and properties as mathematical objects and why they work. Rule-based relations tasks relate with the study of patterns, functions, change and variation (Kaput, 2008; NCTM, 2000). These tasks focus on the relations within a dataset or between datasets and can engage students in identifying relationships, extending, forming and generalizing rules. Different types of generalizations include the factual, contextual and symbolic generalization (Radford, 2003). Known-unknown relations tasks rely on the view of algebra as a cluster of modelling languages (Kaput, 2008) and the problem-solving approach on the introduction to algebra (Bednarz, Kieran, & Lee, 1996). These tasks range from students' opportunities to engage with informal approaches in manipulating the increasing complexity of the relations between known and unknown quantities to those that reveal the power of symbolism in handling unknowns as known, when introduced to algebraic equations, the concept of the unknown, and equation solving.

2.2. Teachers' guidance in curriculum materials

Curriculum materials beyond providing the intended learning opportunities for students, serve also as offering to teachers an image about these learning opportunities. Curriculum materials can help teachers to understand the big mathematical ideas and to think about the development of the content across the years (Ball & Cohen, 1996; Davis & Krajcik, 2005). Materials that are designed to support teachers' learning can help teachers anticipate students' thinking and interpret their responses, support teachers' subject matter knowledge, make more visible the pedagogical intentions, inform about other teachers' approaches, and help teachers see connections between units (Ball & Cohen, 1996; Davis & Krajcik, 2005; Remillard & Bryans, 2004). It should be noted that teachers' experience, beliefs and knowledge play an important role in reading, interpreting and enacting the curriculum materials (Remillard & Bryans, 2004).

Even though these are potential forms of guidance, there are practical limitations with regard to the amount and presentation of guidance (Stylianides, 2007). It is not possible for teachers' guides to make provision for all the interactions that can build students' understanding and all the anticipated students' responses and underlying thinking (McClain, Zhao, Visnovska, & Bowen, 2009). There are also concerns that teachers who use the written guidance without adapting it to the class context or responding to unexpected answers, may lead in less effective learning (Remillard, 2000).

Stylianides (2007) examined the guidance in curriculum materials for proof tasks and categorized the tasks into those that were accompanied by only one solution or by a solution with additional guidance. Additional guidance was considered explanations about why students' engagement in proof task matters, cautious points on how to manage student approaches and discussions that supported teachers' content knowledge.

Based on the above, in this study, four components of guidance for teachers to enact the algebra-related tasks in classroom were examined. The first component was any reference to the mathematical ideas embedded in the task. The second was provision of the expected answer to the task. The third component was the approach to solve the task and the fourth was commentary of how to engage the class with the respective task.

3. Application of the analytic framework

The analytic framework was applied to analyze a textbook series. The process of analysis as well as the methodological decisions are described below in further detail.

3.1. Sample

The sample of this study consists of the curriculum materials used in the Cypriot educational content by the time this study was implemented. In the Cypriot context, teachers used a mandated mathematics textbook series that was published by the Ministry of Education and Culture and introduced gradually during the years 1998–2003. The textbook series consists of the students' textbooks and the teachers' guides for each grade. The particularity of this educational context lies on the fact that all state schools used the textbook series, which was also a unique resource, as no other textbook series was provided by the Ministry to be used by students and teachers in schools. Due to the uniqueness of the textbook series and the limited guidelines in the curriculum document, Cypriot primary teachers depend heavily on textbooks when planning and implementing their lessons (Kyriakides, 1996; Petrou, 2009). Since all teachers work according to the same guidelines, it was meaningful to explore the guidance available to enact the algebra-related opportunities in these curriculum materials.

The student textbook volumes were accessed online from the Ministry of Education and Culture website, while hard copies of the teachers' guides were bought from the Ministry's book warehouse. The sample consisted of 12 volumes of students' textbooks for grades 4 to 6 (four per grade), and three teachers' guides (one per grade).

3.2. Process of analysis

The textbook task was the unit of analysis as it served as a systematic point of reference (e.g. Stylianides, 2007) and the sample consisted of 2,814 tasks. According to Stylianides (2009), a task for the purposes of textbook analysis can refer to "any exercise, problem, activity, or parts thereof that have a separate marker in the students' textbook" (p. 270). As a separate marker, the second level of numbering was used to ensure that there was greater consistency across textbooks, and non-numbered sub segments that were clearly identified in the same way as those on other pages were assigned an external numbering.

Each task in the student textbook was examined to decide whether it matched any of the three kinds of algebra-related tasks, as mentioned above. Drawing on Stylianides' (2009) methodological approach, all the tasks were solved in the same order as they were presented in textbooks. The solving approaches were based on the knowledge of what came earlier in the curriculum to decide about students' expected prior experiences relevant to the task in order to make inferences about what the students were expected to engage with. Also, the available guidance in the teachers' guides was taken into consideration. Based on this, it was possible to decide whether the task could be considered an arithmetically-situated relations tasks, a rule-based relations task or a known-unknown relations task. If the task belonged to one of these kinds of tasks, then it was considered as an algebra-related task.

For each algebra-related task, the respective guidance in the teachers' guides was explored. Along with the first component, the guide was read to identify whether it provided any information about the mathematical ideas embedded in the task by looking at the learning objective of the lesson and specific reference to the task. It was not necessary for the guidance to refer explicitly to algebra in order to consider that there was information about the mathematical ideas embedded in the task. Regarding the second component, it was explored whether the guide provided the correct answer(s) to the task and even whether any anticipated incorrect student answers were included. Based on the third component, it was investigated whether any approaches for solving the task were suggested. Particularly, emphasis was placed to examine if only one approach, or more approaches, or even students' incorrect approaches were mentioned. This decision aimed to make transparent the different levels of guidance considering that algebra has traditionally been seen as part of secondary and higher mathematics, and thus teachers may not possess a comprehensive understanding of the anticipated early algebraic thinking. Finally, following the fourth component, any commentary about how to engage the class with the identified algebra-related tasks was explored such as teachers' questions, images of classroom interaction, and how to organize the students.

The inter-rater agreement was tested by comparing the codes of the first coder with those of a second rater. The second rater coded a subsample of tasks that consisted of three out of the 12 textbook volumes (one volume from each grade). The reliability value related to decisions about whether or not a task in the subsample was algebra-related and the inter-rater agreement was kappa=0.82.

4. Type of guidance and selected cases of tasks

The analysis identified 250 algebra-related tasks out of a total of 2,814 tasks (8.9%) in the textbook volumes for the fourth, fifth and sixth grades. The identification of the algebra-related tasks was based on the definitions of the three kinds of tasks as described above; however, the textbook authors might have had a different definition in mind that could lead to different results. The findings are shown in Table 1 and indicate that information about the mathematical ideas was provided for 86.4% of the algebra-related tasks and the correct answer for 68.8%. The teachers' guides suggested one approach for solving the task for 44% of the algebra-related tasks while for 1.6% of the algebra-related tasks there was guidance for more than one correct approach. Commentary about how to engage the class was found for 15.2% of the algebra-related tasks.

A synthesis of the findings suggests different types of guidance for algebra-related tasks. A more elaborated guidance is regarded when the teachers' guide informed the reader about the mathematical ideas of the task, the correct answer(s), the approaches in solving the task and commented on how to engage the class. This type of guidance was provided for 1.6% of the algebra-related tasks. For 11.6% of the tasks, the guides referred to all the components of guidance under study but provided only one approach for solving the task. A less expanded type of guidance was found for 38% of the algebra-related tasks as the teachers' guide referred to the mathematical ideas, to one suggested approach and to the correct answer(s). A thinner type of guidance that contained information about the mathematical ideas and the correct answer, which was also the most common one in the textbook series analyzed, applied to 68% of the algebra-related tasks.

Table 1
Percentage frequency distribution of guidance in teachers' guides

Components of guidance	**Algebra-related tasks**
Mathematical ideas	86.4
Correct answer(s)	68.8
Suggested approaches	
- One suggested approach	44.0
- More than one suggested approach	1.6
Class engagement	15.2
Types of Guidance	
Mathematical ideas + Correct answer(s) + More than one suggested approach + Class engagement	1.6
Mathematical ideas + Correct answer(s) + One suggested approach + Class engagement	11.6
Mathematical ideas + Correct answer(s) + One suggested approach	38.0
Mathematical ideas + Correct answer(s)	68.0

The results suggest that only a small percentage of the algebra-related tasks were accompanied by the more elaborated type of guidance. Even though limited guidance could raise concerns about how these tasks might be enacted in classroom, it is impossible for all tasks to be accompanied by elaborated guidance. We thus look closer to the tasks to explore how the components of guidance under study appear in the teachers' guides, to discuss what these different types of guidance might mean and what a more elaborated guidance might be.

The task in Figure 1 is from Grade 4 and was categorized as a known-unknown relations task because students were expected to engage in handling non-direct relations between known (number of animals and number of feet) and unknown (number of chicken and number of rabbits) quantities. It is a story problem that cannot be solved with straightforward arithmetical calculations and therefore students need to find other problem-solving strategies, until they will be able to use symbolism to represent and manipulate unknown quantities.

Students' textbook

Theodoros counted the chickens and the rabbits in his farm. He found that all animals were 18. He then counted their feet and found 50. How many chickens and how many rabbits are in Theodoros' farm?

(MEC, Grade 4 Volume B, 1998, p. 125)

Teachers' guide

Learning objective of the lesson: Students would be able to solve problems using different strategies.

Information about the task: The problem could be solved by drawing or using a table. For example, the problem mentions that the chickens and rabbits are 18. Students could draw 18 circles to represent the body of the animals. The teacher could ask students to think whether it is possible for all the animals to be chickens and how they could draw them. How many feet in total? It is anticipated that the students would answer that the feet would have been 36. Afterwards, students are asked to re-read the problem in order to realize that the 36 feet need to be 50. So, they add 2 feet in some animals until they become 50.

Another way is to form a table, similar to the one below:

No. Attempts	**Animals**	**Chickens**	**Rabbits**	**Feet**
1	18	9	9	(9×2)+(9×4)=18+36=54
2	18	10	8	(10×2)+(8×4)=20+32=52
3	18	11	7	(11×2)+(7×4)=22+28=50

Solution: Chickens are 11 and rabbits are 7.

(MEC, Teachers' Book, 1998, p. 91)

Figure 1. Chickens and rabbits task.

It is one of the tasks with more elaborated guidance. The teachers' guide informs about the mathematical ideas embedded in the task, which is in fact developing a mathematical competency that of problem-solving, without any reference to algebra. The correct answer is also provided as well as two approaches for solving the task: (1) drawing the animals, and (2) creating a table. The first approach involves drawing the body of 18 animals, drawing two feet in each animal and then adding two more feet to some animals until the total number of feet is 50. The description of this approach presents questions that the teacher could ask in the classroom, which encourage students to understand the relationship between quantities. Even though it is not a quite elaborate description of how the class could be engaged with the task, it gives an image to the teacher of how it could unfold in the classroom.

The second approach relies on creating a table and attempting to find what might be the number of chickens and rabbits in order to sum up to 18 and calculating the number of feet for these attempts. This approach reminds the guess-and-check strategy. The table shows that in the first attempt, the number of chickens and rabbits is equal. However, the number of feet in total is 54, which is more than 50. Hence, in the second attempt, the number of rabbits (which have double the feet of chickens) is reduced by one and the number of chickens increased by one. It can be seen that the attempts to guess the number of animals are made strategically based on the outcome of previous attempts.

Given that two approaches are suggested to solve the task, teachers create a more comprehensive understanding of this task regarding its enactment in the classroom. Students may solve the task using one of these approaches or the teacher may encourage students to use and compare different approaches. Suggesting only one approach may restrict teachers, who lack the capacity, in discussing different solution approaches and elevating the learning potential of the task. The task below is a story problem with the same structure as the problem in Figure 1. As seen in Figure 2, the teachers' guide suggests solving it only with the guess-and-check strategy.

Students' textbook
215 people attended the concert that was taking place at Lakatamia's theatre. The adults ticket costs 4 euro and the child ticket 1.50 euro. The total amount received was 560 euro. How many adults and how many children attended the concert?
(MEC, Grade 5, Volume B, 1999, p. 57)

Teachers' guide
Learning objective of the lesson: Students would be able to solve problems that involve addition and subtraction of decimal numbers.
Information about the task: Story problem to be solved with the "guess-and-check" strategy.
Solution: 95 adults and 120 children.
(MEC, Teachers' Book, 2002, p. 77)

Figure 2. Theatre task.

The fact that these curriculum materials are the only formal resource available for teachers seems to enhance their authority in planning and enacting mathematics lessons. Since there is no other textbook presentation or formal resource for guidance to provide ground for comparison, the current materials may create an impression (perhaps inadvertently) that there are no other possible or appropriate approaches for implementing the tasks beyond the one suggested. This is the case in the Cypriot context and might be the case also in other small scale and centralized educational systems.

Nonetheless, it is impossible to provide elaborated guidance and multiple approaches for solving each task in the textbook. One way of responding to the dilemma regarding the amount of guidance is indicating to teachers where elaborated guidance for similar tasks can be found in previous pages in the teachers' guides. One could argue that the teachers' guides provided elaborated guidance for the task shown in Figure 1 and hence it was not needed to repeat it for Figure 2. But this argument would not apply for this case, since the two tasks were drawn from different grades. Another way is to present discussions of selected issues separately in the teachers' guides. It is also possible that certain tasks serve different purposes in different lessons. Hence, more clarity would be needed about the purpose of the task, for example if the task serves to practice a specific problem-solving strategy, compare different strategies, link informal approaches with more formal algebraic approaches.

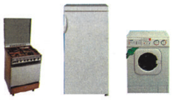
In the tasks in Figure 1 and Figure 2, there was no explicit reference to algebra. It is likely that the authors did not have the intention to exploit the potential of these tasks for students' preparation to algebra. Indeed, the guess-and-check approach does not encourage students to manipulate the quantities in ways they typically encounter in similar problems when using formal algebraic methods. Alternative approaches are to 'check a convenient trial value' to get a sense of the relations hidden in the story problem, and 'denote the as-yet-unknown' by either manipulating the unknown or leaving it unmodified (Mason, Graham, & Johnston-Wilder, 2005). Cai (2004) also described an approach in the Chinese curriculum that relied on identifying the advantages and limitations of different solution strategies and making links between the informal and formal problem-solving strategies. In these ways, students' engagement lays the ground for algebraic problem-solving. If curriculum developers aim to prepare students for algebra, the learning potential of such opportunities could be enhanced and linked with future problem-solving approaches.

For example, the approach 'denote the as-yet-unknown' would have been more purposeful since they could represent the relations between the quantities in an informal way that links the known with the unknown quantities, e.g. Chicken + Rabbits = 18, 2 × Chickens + 4 × Rabbits = 36 and then trying numbers for chickens and rabbits. This approach sets the ground for students' work with formal algebraic problem-solving and provides more coherence in students' learning opportunities with similar opportunities in Grade 4 as seen in Figure 3. As seen in the suggested approach in the teachers' guide, students are expected to denote the unknowns. The task in Figure 3 asks students to find the price for each electrical appliance and the teachers' guide suggests representing the relations in an equation format. Then, by comparing the first two relations, they can find the price of the washing machine and then the price of the other electrical appliances. In this task, there is no suggestion to solve the task by guessing values for each electrical appliance and checking whether the guessed values fulfill the given relations between the known and unknown quantities.

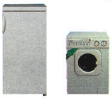
Students' textbook

Read the information provided to find the price of each electrical appliance.

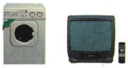
The cooker and the fridge cost 1320 euro.



The cooker, the fridge and the washing machine cost 1800 euro.



The fridge and the washing machine cost 1270.



The washing machine and the television cost 770.

(MEC, Grade 4 Volume C, 1998, p. 63)

Teachers' guide

Learning objective of the lesson: Students would be able to solve problems using different strategies.

Information about the task: It is recommended that students have either cards with the photos of the fridge, washing machine etc. similar those in the book, either cards with the following: cooker, television, fridge, washing machine, 1320, 1800, 1270, 770.

Students could place the cards in the same way as in their textbooks and find the relationships between the electrical appliances.

cooker + fridge = 1320
cooker + fridge + washing machine = 1800
1220 + washing machine = 1800, washing machine = 480
fridge + washing machine = 1270
fridge + 480 = 1270, fridge = 790
washing machine + television = 770
480 + television = 770, television = 290

(The teachers' guide also recommends solving a similar problem to the above).

(MEC, Teachers' Book, 1998, pp. 111-112)

Figure 3. Electrical appliances task.

As mentioned before, the teachers' guides provided information about the mathematical ideas, the correct answer and one suggested approach to solve the task for 37% of the algebra-related tasks. This level of guidance was provided for an arithmetically-situated relations task shown in Figure 4. The task examines the addition and subtraction of odd and even numbers. The guide mentions that students are expected to investigate the general form of these numbers and relates with the objective of the lesson about the notion of the variable. There is no information about how to engage the class with this task, but the guide suggests to solve the task by giving examples of specific numbers. On the one hand, this could be interpreted as a strategy to help students to deal with the concrete before engaging with the abstract nature of the expressions. On the other hand, this also conveys the idea that to generalize the relation between odd and even numbers, all that is needed is to check a few specific numbers. If the second interpretation applies, then the guidance encourages teachers and students to develop empirical arguments, which illustrates inattention on the part of the textbook authors to the guidance provided. Investing on mathematically appropriate approaches is an issue worth being considered by the curriculum developers.

Students' textbook

Complete the tables with the words 'odd' or 'even' to discover the rules for the number operations.

If λ > μ

λ	μ	λ + μ		λ	μ	λ – μ
odd	odd	even		odd	odd	
odd	even			odd	even	
even	even			even	even	
even	odd			even	odd	

(MEC, Grade 6 Volume C, 2000, p. 51)

Teachers' guide

Learning objective of the lesson: Students are expected to use algebraic expressions to represent the relations between numbers and to understand the notion of variable.

Information about the task: Investigation of numbers in general form: odd/number/even number (Requested to give examples of specific numbers).

Solution: odd+even=odd, even+odd=odd, odd+odd=even, odd-even=odd, even-even=even, even+odd=odd.

(MEC, Teachers' Book Volume B, 2003, p. 62)

Figure 4. Odd and even numbers task.

The next task in Figure 5 is a growing pattern task for which the guide provided similar guidance to the task in Figure 4 and it raises interesting issues regarding the information about the mathematical ideas, the correct answer and the suggested approach for solving it. It is a rule-based relations task and provides the graphical representation, a table with the number of cubes and the area of the outer surface for the first six terms. The second question asks students to find the area of 20 cubes glued together as shown below.

According to the answer in the teachers' guide, students are expected to find a general rule in order to be able to find the area based on the number of cubes. This expectation is not stated explicitly but it seems to be indicated by the way the solution to the second question is presented [i.e. (number of cubes × 4) + 2 = 82]. The task also asks students to justify their answer but again a relevant response is not provided in the guide, but the justification is implicitly seen in the equation given (i.e. four sides for each of the cubes plus the additional two sides of the first and the last cube in the row). However, it is not known whether teachers would make similar inferences. The level of explicitness in the presentation of teachers' guides is likely to influence teachers' interpretations and understanding about the expectations of the task.

Students' textbook

1. Complete the table.

Number of cubes	1	2	3	4	5	6
Area of the outer surface	6	10				

2. What is the area of the outer surface of 20 cubes stuck together in a row? Justify your answer.

(MEC, Grade 5, Volume D, 1999, p. 43)

Teachers' guide

Learning objective of the lesson: Students would learn to find the area of the outer surface of cuboids and solve problems that involve the relationship between area and volume.
Information about the task: Relationship between the volume and the area of the outer surface of a cuboid.
Solution (Question 2): (number of cubes × 4) + 2 = 82 square centimeters.

(MEC, Teachers' Book, 2002, p. 148)

Figure 5. Growing pattern task.

The teachers' guide could have mentioned that students were expected to investigate how the pattern grows by looking at the structure of the given figures, and even explain that the task was asking for the number of cubes in the 20th figure so that students identify and generalize a functional relation between the number of cubes and the area of the outer surface. However, in the teachers' guide the information about the mathematical ideas mentioned that the task involved the relationship between the volume and the area of the outer surface and did not refer to the most prevailing processes embedded in the task, that of identifying the structure of co-varying quantities or even generalizing. As seen in the learning objectives, the task was part of a lesson that studied the relationship between the area and the volume. Indeed, these concepts are involved in the task, but these are not at the essence of solving the task. Also, the relationship between the area and the volume in this task applies only for the specific construction of cubes. It is a task in the context of area and volume but engages students in attending to the structure of consecutive figures and finding a rule that links the number of cubes with the area of the outer surface of the figure. The issue raised here is how the tasks are selected to design a lesson in the student textbook. In the design of this textbook series, the textbook authors presented the mathematical topics in an interrelated manner focusing on number operations and properties while topics such as geometry, measurement, probability, and statistics develop simultaneously and are not contained within separate lessons (Petrou, 2009). However, such presentation is more likely to make it difficult for teachers to track the development of mathematical topics across years or perceive how tasks serve the goals of primary mathematics. If curriculum materials are supposed to also educate teachers, then they need to help teachers understand the development of content and consider the tasks in the context of the larger curricular picture (Ball & Cohen, 1996). It is purposeful for teachers to know the main ideas and foci of the lesson and the rationale behind the selection of tasks and how these are linked together. Another option in the design of lessons would be to have separate lessons for different topics, i.e. a whole lesson on growing patterns.

Furthermore, the issue discussed above regarding the approaches for solving the task is again raised for the growing pattern task. The teachers' guide shows only one way for reaching the answer but there are various ways that students could find the 20th figure. For example: (a) by multiplying 18 cubes times four sides and adding 2 cubes times five sides; (b) by multiplying 20 cubes times six sides and subtracting 18 cubes times 2 sides and then subtracting two more sides from the first and the last cube. Different perspectives are derived from attending to which elements remain constant and which elements change in a predictable and consistent manner (Andrews, 2002). The guidance about the class engagement could have mentioned important questions such as 'which elements change and which ones remain the same?'. The use of sign drawings, sign digits, as well as speech and gestures are also means for constructing meaning in mathematical generalizing processes, which are referred by Radford (2003) as the semiotic means of objectification and lead to differences in types of generalization.

The tasks discussed above could neither have been considered straightforward nor trivial and providing guidance for teachers is more meaningful for such tasks. An explanation that may apply for the 68% of algebra-related tasks, which were not accompanied by a suggested approach or guidance about the class engagement, is that these were routine tasks or tasks for which the approach was straightforward and hence an elaborated form of guidance would not have had an additive role. For example, the task in Figure 6 below is an arithmetically-situated relations task, in which students engage with the symbolic and verbal generalization of the multiplicative identity property. It belongs to the tasks for which a thinner type of guidance was provided with information about the mathematical ideas and the correct answer only. The presentation of the task does not provide context for much elaboration. Hence, both the design and demand of the task as well as what have preceded the task might determine the need of guidance that would be meaningful for teachers to enact the task in classroom.

Students' textbook
Find the products and write your comments.

$86 \cdot 1=$	$\kappa \cdot 1=$	Comments: ______________________
$754 \cdot 1=$	$\lambda \cdot 1=$	________________________________
$1 \cdot 46{,}3=$	$1 \cdot \mu=$	________________________________
$1 \cdot 626{,}43=$	$1 \cdot \nu=$	

(MEC, Grade 6 Volume C, 2000, p. 57)

Teachers' guide
Learning objective of the lesson: Students would recognize the identity property of addition and multiplication.
Information about the task: Investigation of the multiplicative identity property using numbers and symbols.
Solution: The product of any number with one is equal to the number.
(MEC, Teachers' Book Volume B, 2003, pp. 70-71)

Figure 6. Identity property task.

The findings also showed that this textbook series did not provide any guidance about possible incorrect student approaches and answers. One reason maybe that this was not an intended approach by the textbook authors while another reason maybe that the literature was not so comprehensive at the time these textbooks were published. However, since nowadays the field has developed substantial knowledge about students' difficulties with algebra-related topics, it would be educative for teachers to access relevant findings in the teachers' guides. For example, regarding the growing pattern task in Figure 5, among students' generalization strategies are the recursive strategy and the whole-object strategy (Lannin, 2005). Students who employ the recursive strategy build on the previous terms to find the next term. These students would have attempted to find the area for the 7th up to the 19th figure, in order to be able to find the 20th figure. Another strategy is the whole-object that could lead to incorrect solutions. Students use a unit and multiply it to find a larger unit. For example, students could have argued that since for two cubes the area is 10 square centimeters, then for 20 cubes the area would have been 200 square centimeters.

The analysis of teachers' guides showed variation in the amount of guidance and in the treatment of the components of guidance under study. Teachers do not seem to have comprehensive guidance in a systematic way for the implementation of algebra-related tasks in classroom. Considering that algebra is a topic that has gained increasing emphasis in primary mathematics, curriculum developers would need to reconsider what might be the nature of the guidelines that could help teachers enhance students' engagement with relevant learning opportunities. Curriculum materials have the potential to offer opportunities for teachers' learning (e.g. Collopy, 2003), which is particularly important for algebra-related tasks since they have not been traditionally considered part of primary school mathematics.

5. Conclusions

This paper presented an analytic framework for examining the guidance in teachers' guide to enact algebra-related tasks and discussed the findings from applying the framework to analyze the textbook series used in the Cypriot educational context. The findings provided the context to discuss aspects of the guidance that seem necessary and meaningful for teachers. In this way, the paper contributes in beginning the discussion of how the necessary guidance for teachers to implement algebra-related tasks might look like and in developing understanding towards this direction. The elements of guidance and the issues discussed above could also inform the designers of curriculum materials. Three main issues were revealed in the presentation and discussion of the selected cases of tasks.

One of the issues was the amount of guidance, which has also been raised in other studies (e.g. Davis & Krajcik, 2005; Stylianides, 2007). The textbook series did not provide systematically comprehensive guidance for teachers. It could have been supportive, at least to beginning teachers, if the teachers' guide provided all the components of guidance under study – reference to the mathematical ideas, the correct answer(s), approaches to solve the task, and commentary about the class engagement. But this would be an unmanageable and impractical approach both for curriculum developers and for teachers. As mentioned above, the amount of guidance that would be meaningful for teachers seems to depend on the nature of the task, on what have preceded the task and on how the teachers' guide is organized. It should not be overlooked that most teachers might not have the time to read extensive guides and also prescriptive guidance that ignores teachers' autonomy may result in less effective curriculum materials (Davis & Krajcik, 2005).

A second issue is the level of explicitness in the presentation of the aspects that form the guidance. Beyond the tasks for which no information was given about the mathematical ideas, the expectations of algebra-related tasks were not presented in a clear-cut manner. It is critical for teachers to be aware of the mathematical ideas embedded in the tasks, which is considered more important than guiding their actions (Remillard, 2000). Given the fact that algebra has traditionally been considered a mathematical topic for secondary school, there is a danger that the potential of these tasks to engage students with early algebraic ideas will not be fulfilled. Teachers' approaches to tasks are underlain by the different ways they read the textbooks, which in turn are influenced by their beliefs about teaching and their expectations of students' learning (Remillard, 1999). Therefore, by not providing explicit information about the role of these tasks, textbooks allow further space for disparate interpretations among teachers and thus more variability in the opportunities that teachers offer to students to engage with algebra-related topics. This is particularly important considering the findings that primary teachers have rather narrow conceptions about algebra-related tasks and a rather limited understanding of the learning potentials of these tasks (Chick & Harris, 2007; Stephens, 2008). Hence, concerns are raised regarding the implementation of algebra-related tasks in primary school classrooms.

A third issue is the selection of approaches. The opportunities for students to engage with algebra-related tasks should be presented in developing progressive steps towards students' preparation for algebra, if this is an intended goal by the textbook authors. Also, the approaches for solving a task suggested in the guides need to be mathematically appropriate. The provision of presenting different possible students' approaches expands teachers' repertoire and knowledge base. This is likely to enhance the classroom discourse as teachers may be more prepared to adapt flexibly to students' answers. The teachers' guide could contribute towards supporting teachers, but it should not be seen as an authoritative resource or even a universal remedy. The written guidance is only one approach to provide support that should not be over-estimated, and it should not be expected that all teachers have to follow the guidance. A teachers' guide should be written in a way that considers teachers' agency and the need for teachers to make decisions and adaptations.

Further research is needed to explore how teachers interpret the available guidance and how they enact the algebra-related tasks in the classroom. The field needs more studies on what type, amount and form of guidance is rather optimal for teachers to provide sense-making opportunities that prepare students for algebra. Based on the findings of this study, it is of interest to explore further the guidance for different tasks according to their cognitive demand and their place in the curriculum materials. More understanding could then be developed about how the written guidance influences the classroom practices and what kind of presentation format would be meaningful and practical for teachers.

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