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A formula for an objective measurement of students' understanding difficulties of a mathematical text. Evaluative and educational use.

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Sunto. In questo articolo si fornisce una formula oggettiva per la valutazione empirica della comprensione di un testo di matematica da parte degli studenti di qualsiasi livello scolastico. Di tale formula si suggerisce un uso valutativo e didattico.

Abstract. This article provides an objective formula for the empirical evaluation of the understanding of a mathematical text by students of any level of education. Of this formula we suggest an evaluative and educational use.

Resumen. Este artículo proporciona una fórmula objetiva para la evaluación empírica de la comprensión de un texto matemático por parte de los estudiantes de todos los niveles. De esta fórmula se sugiere un uso evaluativo y un uso educativo.

Parole chiave: formula di leggibilità, formula di comprensione, difficoltà di comprensione di un testo matematico, leggere di matematica, scrivere di matematica (TEP)

Keywords: readability formula, formula of comprehension, comprehension difficulties of a mathematical text, read mathematics, writing mathematics (TEP)

Palabras clave: fórmula de legibilidad, fórmula de entendimiento, dificultad de comprensión de un texto matemático, leer la matemática, escribir de matemática (TEP).

1. Historical antecedents and reference framework

In the 80's the so-called "readability tests" were studied a lot. They were born in the previous decades, mainly regarding mathematics. This new generation of studies lasted for almost fifteen years before coming to an end but the issue of student's understanding of mathematical texts persists… More than a colleague who teaches mathematics in whatever school level claims that reading and understanding a mathematical text is in general an arduous endeavour on the part of the student.

Among all tests, the "cloze texts" were widely spread. A cloze test consists of a mathematical text with certain words removed, one every n (the n-th, the $2n$-th, so on and so forth); the student is invited to read the text with deleted terms and replace the missing words with the word he believes more appropriate to restore the meaning of the text he is reading.

Several empirical trials, mainly in English and French but also in Italian and Greek, brought to certain choices regarding constants and variables that appeared in the so-called readability formulae.

Amongst the several researches in this respect we highlight below the ones that offered more instruments to ours. We will show our research later in this paper.

In various disciplines, not necessarily mathematics, the researches of Vogel and Washburne (1928) that are classic in this field arise for importance among others. Their analysis is based on various factors that focus on the length of the passage one is reading and on the orthographic typology/difficulty of the erased words. Instead the formula we are looking for should measure the features of the difficulties, therefore irrespective of the length of the passage. Their research refers to the object "mathematics text" (more or less "easy" from an objective point of view), ours refers to the person "student who reads the mathematics text" (the student who finds it more or less difficult to reconstruct the initial text).

There are also the studies of Taylor (1953), Rankin (1970) and Henry (1973); although there are many specific differences they have much in common. All the aforementioned researches claim the opportunity and validity of cloze tests as an instrument of analysis and provide interesting results taken from empirical investigations carried out in the classroom. The analyses they present is so convincing that we also decided to choose a formula based on cloze tests.

Shifting only to mathematics, we point out the research we reckon really specific for our discipline: Kane, Byrne and Hater (1974). This research carried out on English mathematics texts, provides results related not only to young students' skills in understanding a mathematics text, but it analyses ways that can help students with difficulties to understand texts. We will see how this research somehow anticipates ours that develops in the same direction.

Amongst other researches, we have been influenced by the researches of Gagatsis (1980, 1982, 1984, 1985, 1995) and Gagatsis and Chaney (1983).

In chronological order we consider the last one (Gagatsis, 1995) because it analyses various formulae and results included in previous studies. The formulae that are compared are all of the type:

$$\text{Readability} = ax+by+cz+k$$

where a, b, c, k are numeric constants assigned according to experience, determined by the language and other factors, while x, y, z are variables that represent objective measurable elements of the tests.

The formulae developed by Gagtsis are based on classic previous ones. For example Rudolph Flesh (1951) formula for English language is the most spread:

$$\text{Reading ease} = 206.835\text{-}0.846x\text{-}1.015y$$

where x is the average number of syllables contained in 100 words and y is the average number of words per sentence.

A famous example, very often quoted as an adjustment to French, is that of Kandel and Moles (1958):

$$\text{Reading ease} = 206.85\text{-}0.74x\text{-}1.015y;$$

whereas an adjustment to Greek appears in Gagatsis (1982):

$$\text{Reading ease} = 206.85\text{-}0.59x\text{-}1.015\text{y}.$$

In Gagatsis (1995) we find a formula of the same type applied to an Italian text devised for 8^{th} grade middle school students (the text in Gagatsis's work was taken from D'Amore, 1981).

Applying to this Italian excerpt the readability formulae determined heuristically for French and Greek respectively, we obtain two very different values, 19.05 e 52.55 respectively. This result proves that the formula is specific for a language and its implicit lexicographic variables (Gagatsis,

1995). Our formula instead, based on the person who reads, has the tendency to be not strictly bound to the language but to the typology of the deleted words that must be restored in their correct position.
On the basis of this experience, Gagatsis (1995) shows how we can empirically construct a readability formula, highlighting that its functionality requires a much higher number of constants and variables. Nevertheless we remain in the domain of determining the greater or smaller difficulty in text reading, thanks to cloze formulae (i.e. restoring words after the every *n* deletions) that we described above. But the need of more elements makes the formula almost not applicable for purposes that would like to have a methodological and didactical character and to be concrete and useful for real classroom situations (as it was proved by studies realized during a maestría thesis in Educación Matemática at the Unversitiy of Medellin; the authors were supervisors of this thesis on which we will focus later).

2. Students' difficulties in managing mathematical language

Researches in this field further highlighted a well-known widespread difficulty of the student that is linked in general to the interactions between mathematical language and common language. Various authors studied this topic, for example Laborde (1982, 1990, 1995).
This Author in many occasions highlighted the following factor: the mathematical discourse has its own semiologic code that entails the economy of the expression obtained, due to two essential components specific of the mathematical language: a designation function and a localization function. This implies a univocal and sometimes new semantic use of the terms and the expressive modalities of natural language terms that are somehow made specific thanks to their reinterpretation in the mathematical discourse.
Another characteristic of mathematical discourse is its universality (Laborde, 1995).
Furthermore within mathematical discourse we testify a blending of natural language and a more or less widespread use of symbolic writing. This phenomenon does not occur in the natural discourse.
The student and the teacher resort to three types of strategies to face the remarkable difference between the two languages:
(1) repetition throughout the text of the expression that describes the mathematical object (a circumstance to avoid as much as possible in natural language: repetitions are usually considered improper in literary texts and necessary in the mathematical ones);
(2) reference to temporal characteristics that make sense in the natural language but not in the mathematical one;
(3) use of distinction properties, typical of natural language but not of the mathematical one, for example the ones relative to the spatial dislocation, amplitude, or time (Laborde, 1995).
These considerations have led Laborde to analyse reading difficulties of mathematical texts since in them natural language and mathematical language are complementary and intertwined; we refer the reader to Laborde (1995).
A detailed analysis of this issue is carried out in D'Amore (1999, all of chap. 8). But more specific works that highlight problems of linguistic interpretation both in reading in general, and mainly in the reading and interpretation regarding written texts of verbal problems are in D'Amore (1993a, 1993b, 2014). They show the arising of a real conflict between the two languages. This occurs also when the teacher has set out a-didactical situations that bestow on the pupil a leading role in the construction of his knowledge, language included (as a result of the devolution) (D'Amore, Sandri, 1994). (For specific terminology we refer the reader to D'Amore, 1999).

Other very interesting studies focussing on the interferences between mathematical language and common language are in Maier (1989b, 1996). They show several cases of missed communication

on the part of the student, missed communication that could be trivially ascribed to mathematical ignorance but it is much better explained by resorting to such interferences.
All these studies agree, despite specific differences, on the following points:
(1) they all highlight student's difficulties in conquering the characteristics of mathematical language (D'Amore, 1999);
(2) in a more or less explicit and aware form they study the feeling, experienced by the student (of any age), that he must use, to accomplish the didactical contract, a supposed characteristic of what he perceives as the mathematical language used in the classroom by the teacher and by his peers who are successful in assessments; this entails the creation of a specific language, most often inappropriate, that has been ironically termed as "matematichese"[1] (D'Amore, 1987).
An iterated analysis of these difficulties, carried out from different angles and collecting the instances pointed out by the previous Authors can be found in D'Amore (1993b, 1996, 1997, 2000), D'Amore and Martini (1998) and in D'Amore and Sandri (1994, 1996).
The scientific community has developed a broad discourse regarding the difficulties in interpreting and handling a discourse that is not rigidly formal and mathematical but rather interweaves natural language and school mathematics. The problem we still have to face is to establish a formula as much as possible objective for a quantitative-numeric evaluation of such interpretation difficulties – it is better to say interpretation difficulties of such a text during reading.

3. Towards an objective formula for evaluating student's interpretation difficulties of a mathematical text.

We have recently proposed again the study and implementation of the classical readability/easiness formulae of a mathematics text in Colombia as a Master degree thesis in Mathematics Education on the understanding of mathematical texts in grades 6 and 7 carried out by two students of the University of Medellin.[2] We have all realized, supervisors and master students, that each one of the readability formulae we used was artificial, complicated to handle, difficult to apply and interpret in concrete.
Moreover its interpretation was relative to the text and not to the students' difficulties thereby making uncertain the subsequent articulation of a didactical intervention focussed on the students.
We found it necessary to devise a formula valid for all school levels oriented towards the single student rather than towards the object of study itself, accomplishing also the following three definitions:
(1) What does it mean, in this context, "mathematics text"?
(2) What do we mean when we claim that a given text is suitable for a specific class of students?
(3) What does it mean to "understand" a passage of a mathematics text?
(4) Which variables determine the differences among the different mathematics texts?

(1) In our understanding, a mathematics text is a text with a mathematical feature or a mathematical content, not necessarily a school textbook. It can be an extra-curricular text or a reading book with a feature or a content that can be considered mathematical.
(2) A text is suitable for an *n* grade class (for example grade 5) if it is unanimously considered as such by the author, the publisher, and the teacher who chooses or advises it.

[1] The term has been introduced to resemble the Italian "burocratese", conceived at the end of the '70s with a derogative connotation against the worthless, complicated, and incomprehensible language used in Italian bureaucracy.
[2] We are referring to professors Angélica Molano Zárate and Clara Cecilia Rivera Escobar who in 2013 defended together their thesis *El lenguaje narrativo como propuesta didáctica para provechar los obstáculos de la comprensión en contexto matemático* at the University of Medellin, under the supervision of Bruno D'Amore and Martha Isabel Fandiño Pinilla, earning the Maestría in Educación Matemática degree.

(3) Following the tradition of readability and comprehension that dates back to the formulae quoted in 1., we consider that a student has understood a mathematics passage suitable for his class if he is able to "cloze" the passage that has deleted parts, i.e. to choose successfully the missing words in the passage replacing them with the exact erased term or a correct synonym according to the judgment of an expert who checks the test or of the class teacher. At this point a problem arises immediately: What does it mean "successfully"? We believe that the formula we have worked out, proposed, tested, modified several times on the basis of heuristic trials, and that we will show soon in this paper – measures a numeric scale of greater or lesser comprehension of the passage on the part of a student, ranking from a minimum value (none of the missing words has been recognized and replaced correctly), to a maximum value (all the deleted words have been recognized and replaced correctly, exactly the deleted words or their synonyms).
(4) The variables that have been chosen to single out the differences regarding the different types of mathematics texts are the ones described in (1) but also the following: texts that include figures in written passages, texts that may or not include formulae (the formula can be made also of one single letter or an operation symbol, or something more complicated based on them).

In this paper we deal with the measurement, as much objective as possible, of the viable missed comprehension of a mathematical text by a student. But we do not deal with the causes of the missed comprehension. They can have profound explanations based on different theories that we do not explore. They can stem from conflicts between cultures, lack of specific lexical training, trivial mathematical ignorance, inability of handling semantic coordination, inadequacy of the information available, misunderstanding of the task, semiotic obstacles etc. We believe that it is possible and necessary investigating in this direction, but it requires as a basis a sort of measurement of the true incomprehension of a mathematical text, not a vague impression on the part of the teacher. This is the reason why we decided to face at first the issue of measurement.

4. Choices regarding the passage submitted to the comprehension formula, objective of the research and methodology

After various attempts we decided to delete only the words (even if technical or specific) disregarding the formulae. The formulae are left unchanged and they are not considered among the words. Also isolated symbols such as letters, numbers, operation signs etc. are considered as formulae.

Example from grade 3, primary school. In the experimentation the texts that follow were given in Italian, but we propose here their English translation.
One of the complete texts is the following (Paciotti, Volpati, Meloni, 1978, p. 185-185):

> With the ten digits that you studied during the past years we can construct all the numbers we want, to infinity. According to the place value of the digits, numbers acquire a different value. Each grouping of ten objects makes up a ten, written 10. If a number is made up of two digits the right digit represents the units (u), the left digit the tens (ts).
> 28 = 2 tens, 8 units
> A group of 100 objects, i.e. one hundred units (100 u), forms a hundreds written 100. In a hundreds there are ten groups of ten objects, i.e. ten tens.
> If a number consists of three digits, the right digit represents the units, the middle one the tens, and the left one the hundreds (h)
> 289 = 2 hundreds, 8 tens, 9 units.

On the basis of our choices, we delete the words in place 5,10,15 and so on, counting only the words, neither punctuation signs, nor letters forming a formula, nor numbers. Instead of the deleted words there is an empty space in which the student has to write the missing word or a synonym to "cloze" the test giving it a meaning. Here is the text we proposed to the students:

With the ten digits … you studied during the … years we can construct … the numbers we want, … infinity. According to the … value of the digits, … acquire a different value. … group of ten objects … up a ten, written 10. … a number is made … two digits the right … represents the units (u), the … digit the tens (te).
28 = 2 tens, 8 …
A group of 100 objects, … one hundred units (100 u), forms … hundreds, written 100. In a … there are ten groups … ten objects, i.e. ten …
If a number consists … three digits, the right … represents the units, the … one the tens, and … left one the hundreds (h)
289 = 2 …, 8 tens, 9 units.

The deleted words are therefore:

that, past, all, to, place, numbers, Each, makes, If, of, digit, left, units, i.e., a, hundreds, of, tens, of, digit, middle, the, hundreds.

We show below the original Italian text: the complete text and the text with the deleted words respectively.

Con le dieci cifre che hai studiato gli anni scorsi, possiamo costruire tutti i numeri che vogliamo, all'infinito. Secondo il posto occupato dalle cifre, i numeri acquistano un valore diverso. Ogni raggruppamento di dieci oggetti forma una decina e si scrive 10. Se un numero è composto da due cifre, la cifra a destra rappresenta le unità (u), quella a sinistra le decine (da).
28 = 2 decine, 8 unità.
Un gruppo di 100 oggetti, cioè cento unità (100 u), forma un centinaio e si scrive 100. Nel centinaio ci sono dieci gruppi di dieci oggetti, cioè dieci decine.
Se un numero è composto da tre cifre, la cifra a destra rappresenta le unità, quella in mezzo le decine e quella a sinistra le centinaia (h).
289 = 2 centinaia, 8 decine, 9 unità.
Con le dieci cifre ⋯ hai studiato gli anni ⋯ , possiamo costruire tutti i ⋯ che vogliamo, all'infinito. ⋯ il posto occupato dalle ⋯ , i numeri acquistano un ⋯ diverso. Ogni raggruppamento di ⋯ oggetti forma una decina ⋯ si scrive 10. Se un ⋯ è composto da due ⋯ , la cifra a destra ⋯ le unità (u), quella a ⋯ le decine (da).
28 = 2 decine, 8 unità.
⋯ gruppo di 100 oggetti, cioè ⋯ unità (100 u), forma un centinaio ⋯ si scrive 100. Nel centinaio ⋯ sono dieci gruppi di ⋯ oggetti, cioè dieci decine.
⋯ un numero è composto ⋯ tre cifre, la cifra ⋯ destra rappresenta le unità, ⋯ in mezzo le decine ⋯ quella a sinistra le ⋯ (h).
289 = 2 centinaia, 8 decine, 9 unità.

Therefore the deleted words are:
che, scorsi, numeri, Secondo, cifre, valore, dieci, e, numero, cifre, rappresenta, sinistra, Un, cento, e, ci, dieci, Se, da, a, quella, e, centinaia.

An example from the final year of secondary high school.
One of the chosen complete texts is the following (Bagni, 1996, p. 1314):

Let us consider a real function f of real variable x denoted by $y=f(x)$ and the point of abscissa $x=c$ belonging to its domain. The direct evaluation of f in correspondence to the abscissa $x=c$ traditionally describes the behaviour of the function at the point of abscissa $x=c$.
The traditional method to evaluate the function at point $x=c$ is condensed therefore in the phrase:
«*in formula y=f(x), substituting the variable x with the value x=c, we obtain for y the value f(c)*»
which is equivalent to claiming that:
«*at the point of abscissa x=c, the function is f(c)*».
But it is important to stress that from such a procedure we obtain information regarding only the behaviour of the function f at the single point of abscissa $x=c$. This information fails to accomplish the need of knowledge in a broader «zone» pinpointed by the abscissa $x=c$.

Here is the text with the deleted words:

Let us consider a … function *f* of real variable *x* … by *y=f(x)* and the point … abscissa *x=c* belonging to its domain. … direct evaluation of *f* in … to the abscissa *x=c* traditionally … the behaviour of the … at the point of … *x=c*.
The traditional method to … the function at point *x=c* … condensed therefore in the …:
«*in formula y=f(x), substituting the … x with the value x=c, we … for y the value f(c)*»
which … equivalent to claiming that:
«*… the point of abscissa x=c, … function is f(c)*».
But it … important to stress that … such a procedure we … information regarding only the … of the function *f* at … single point of abscissa *x=c*. … information fails to accomplish … need of knowledge in … broader «zone» pinpointed by …abscissa *x=c*.

We show below the original Italian text: the complete text and the text with the deleted words respectively.

Consideriamo la funzione reale *f* di variabile reale *x* espressa da *y*=f(*x*) ed il punto di ascissa *x=c* appartenente al dominio di essa. La valutazione diretta della *f* in corrispondenza dell'ascissa *x=c* descrive, tradizionalmente, il comportamento della funzione data nel punto di ascissa *x=c*.
Il tradizionale metodo per valutare la funzione nel punto *x=c* si condensa quindi nella frase:
«*nella formula y=f(x), sostituendo alla variabile x il valore x=c, otteniamo per la y il valore f(c)*»
e ciò equivale ad affermare che:
«*nel punto di ascissa x=c, la funzione f assume il valore f(c)*».
È però importante sottolineare che con un procedimento del genere otteniamo un'informazione riguardante esclusivamente il comportamento della funzione *f* nel singolo punto di ascissa *x=c*: e non sempre questa informazione è in grado di esaurire la conoscenza della funzione in una più ampia «zona» individuata dall'ascissa *x=c*.
Consideriamo la funzione reale *f* ⋯ variabile reale *x* espressa da *y*=f(*x*) ⋯ il punto di ascissa *x=c* ⋯ al dominio di essa. ⋯ valutazione diretta della *f* in ⋯ dell'ascissa *x=c* descrive, tradizionalmente, ⋯ comportamento della funzione data ⋯ punto di ascissa *x=c*.
Il ⋯ metodo per valutare la ⋯ nel punto *x=c* si condensa ⋯ nella frase:
«*nella formula y=f(x), ⋯ alla variabile x il valore x=c, ⋯ per la y il valore f(c)*»
⋯ ciò equivale ad affermare ⋯ :
«*nel punto di ascissa x=c, ⋯ funzione f assume il valore f(c)*».
⋯ però importante sottolineare che ⋯ un procedimento del genere ⋯ un'informazione riguardante esclusivamente ⋯ comportamento della funzione *f* nel ⋯ punto di ascissa *x=c*: e ⋯ sempre questa informazione è ⋯ grado di esaurire la ⋯ della funzione in una ⋯ ampia «zona» individuata dall' ⋯ *x=c*.

Obviously the typology of words is very variable. In the first example there are technical terms of mathematics (numbers, digits, ten, number, hundreds), language words with a logical character or intrinsically descriptive (that, second, left, right), words of the natural language (past, value, a, from, to, that). This occurs also in the second example.
The experience that stems from bibliography and heuristic tests repeated several times proves that the typology of the words affect the comprehension of the text. It proves that words of natural language are more recognizable than the words with a logical character and technical terms of mathematics. Therefore when evaluating a student's difficulty in restoring a passage taken from a text suitable for him, the incidence of the mistake has to be "weighed". In fact not recognizing the word "digit" has a different incidence from not recognizing the word "and", as well as it has a different incidence from not recognizing the word "from".
We can eventually specify the objective of this research: to create a "closure test" formula, school level independent, that measures the student's difficulty facing the comprehension of a passage taken from a mathematics text suitable for his school level.

The student is exposed to a test that consists in giving him a text after the deletions specified above, inviting him to replace the empty deleted spaces with the words he reckons missing, in order to give meaning to the passage.

The formula takes into account all the variables involved including, as mentioned above, the typologies of the erased words.

The research adopted an empirical methodology. We started from a formula chosen amongst the ones handed over to history by bibliography. We then carried out preliminary investigations to probe the goodness of the formula, adjusting the coefficients until we arrived to the present ones. According to the results, on the basis of the relationship between the values of the established indexes and the comprehension indexes, we gradually adjusted the indexes until we recognized as satisfactory the present ones (that we will see soon).
For example, only after the inspection of the various passages, at different school levels, we decided not to delete the formulae (interpreted in a broad sense as we said before) and to delete only the words. Indeed the deletion of formulae was at the same time:
a) completely random according to the selected passage;
b) too much significant in many passages and difficult to reconstruct by a novice.
On the other hand, we felt that, in order to evaluate the textual comprehension, it would have been interesting and reasonable to compare the difficulty in establishing a relationship between natural language and the formal one rather than focussing on individual mathematical competencies (according to what we already said above, following the researches of Laborde, D'Amore and Maier).

We carried out the tests mainly in Italy, thanks to the collaboration of various teachers, members of the RSDDM research group of Bologna, involving primary and secondary schools.
On fixed days, the teachers delivered to their students and to students of parallel classes the deleted passages we described above, they explained the task to the students, and gave them plenty of time to carry out the test.
The experimentation involved 25 primary school classes, 4 middle school classes (2 in Italy and 2 in Colombia) and 10 high school classes, for a total of 39 classes. The test has been carried out by 463 primary school students, 66 middle school students and 127 high school students, for a total of 656 pupils (but many of them carried out more than one test).

5. Our proposal for a formula of pupils' comprehension of a mathematical test.

Let us consider a passage T that contains n words. We do not consider formulae or punctuation signs.
The number of deleted words is Int(n/5) i.e. the integer part of the rational number $n/5$.
The deleted words belong to the following categories:
c_1) the words of the current language without a logical nor a technical character (number a);
c_2) technical words of mathematics (number b);
c_3) language words with a logical character (connectives: not, and, or, implication, …; quantifiers: none, some, all, …; deductive: given that, because, prove, …) (number c).
Therefore: $a+b+c=\text{Int}(n/5)$.
Note that n, Int(n/5), a, b, c are all natural numbers.
Let:

$$m_T = a\times 0.1 + b\times 0.3 + c\times 0.4;$$

the rational number m_T is termed as "T difficulty index".
Let:
a' be type c_1 words that the subject S correctly recognizes ($a'\leq a$);
b' be type c_2 words that the subject S correctly recognizes ($b'\leq b$);
c' be type c_3 words that the subject S correctly recognizes ($c'\leq c$);
Consider the formula:

$$r_{TS} = (a\text{-}a')\times 0.1 + (b\text{-}b')\times 0.3 + (c\text{-}c')\times 0.4;$$

the rational number r_{TS} is called the "T comprehension index on the part of S".
If $r_{TS}=0$, we consider the comprehension on the part of S of the passage T perfect;
If $0\leq r_{TS}<m_T/2$ we consider the comprehension of the passage acceptable or positive;
If $m_T/2\leq r_{TS}\leq m_T$ we consider the comprehension of the passage insufficient or negative.

If we enter the formula in Excel and insert the number of mistakes, the value of the single r_{TS} is given automatically simplifying calculations significantly. The evaluation of the scores of r_{TS} is immediate.

6. A specific note regarding the tests we carried out to assign the previous empirical values

Thanks to the collaboration of the members of the research group and other isolated available teachers we have carried out many empirical tests with different values, choosing variable components that play a prominent role in this situation:
(1) students who are judged to be more or less reliable by teachers,
(2) mathematical texts objectively very simple or vice versa very difficult,
(3) deleting also formulae or part of them,
(4) giving in a class a text that had been judged suitable for a previous class,
…
In other words, we have set and analysed several variables to end up with the values 0.1 , 0.3 , 0.4 mentioned above (that sometimes substantially change the ones that appear in classic formulae).
We decide not to report here specifically all the tests we carried out and all the preliminary results we obtained. They are anyway available to the researchers, if they consider them useful or interesting for the continuation of these studies.
Many of our texts are tabulated and are therefore available as Excel files.
Amongst the empirical tests that provided interesting hints about the suitable value to use, we found particularly useful what we called the "simplified test" (see point 4 above).
Suppose that a student Sq obtains a score r_{T1Sq} in the reading of a passage T1 suitable for his school level, and that he obtains a value r_{T2Sq} in the reading of a passage T2 suitable for a class of a previous school level to the one Sq belongs to. The formula has an acceptable empirical sense if in general $r_{T1St}>r_{T2St}$.
The study of comprehension results carried out on simplified tests was extremely helpful in identifying the empirical values we have chosen.

7. Results of the research

We highlight two features that have to be examined.

7.1. The formula objectively expresses the comprehension of a given text under consideration. Therefore it provides an empirical criterion for evaluating the comprehension of a text *per se*. Based on the choices we made, such an evaluation refers to students who belong to the school level for which the text has been conceived, written, published and chosen.

7.2. Once we have carried out a test in the classroom on a text T on n students S1-Sn, we can accomplish a statistical survey on the n values r_{TS} we obtained and see, for example, where the student Sp is positioned ($1\leq p\leq n$). This ranking within $1\leq p\leq n$ points out the objective difficulty of Sp compared with his classmates.

This type of analysis is not present in the previous works listed in the References, but the colleagues who took part in the research have explicitly required it.
Indeed more than a colleague, after repeating the test several times with his regular class, was surprised of the objective difficulty encountered by some students in proposing the correct words to replace the empty spaces in order to make sense of the text considered by the teacher easy to interpret. So to speak, they stressed the need to foster the textual comprehension on the part of the students involved in the reading of mathematical passages.
We report the words that a colleague-researcher has sent to us in written form:

> «(…) on the other hand, when did we ever leave students alone reading a text? Never, we are always ready to explain and interpret for them. Even when we give the text of a problem, we explain it word by word. The student never interprets a text, he has to listen only to our interpretation (…)».

She concludes:

> «Taking part in this research I have understood that I have to risk, give my students the opportunity to read by themselves, even if they make mistakes, they must learn to understand the meaning by themselves (…)».

(The term "to risk" echoes very famous words of Guy Brousseau …).
This kind of reflections enhances what already emerged in some of the previous researches regarding students' reading and comprehension abilities of a mathematics text at any age. (D'Amore, 1996). And an empirical research is transformed in an unexpected didactical challenge that seems very appropriate.

There are also two variables that have been studied but not scrutinized and that we want to highlight for their peculiarities:

7.3. Variable one: the presence of an explicative figure in text T that is being used as a reading in the test.
In some texts a figure appeared as part of the text or along with the text, only when the content of the passage had a geometrical character. As it always happens in these cases, the drawing explains in the figural semiotic register (or, more in general, in a non discursive register) what the text says in the common language (in his mathematical version). But, as it is well known, many teachers are sure that a double register helps the student's textual comprehension even if in many occasions, for example in the works quoted below, there is evidence that it is not necessarily so. The so-called "visualization" sometimes complicates comprehension as it has been shown in research articles [D'Amore (1995), Bagni (1997, 1998), see also articles in D'Amore et al. (1999)].
In other words, some teachers, mainly in primary and middle school, thought before the test that the presence of an explicative figure in text T would have clearly improved the value of r_{TS}. This event did not simply occur.
We believe that this point requires further research based also on specific topics regarding semiotics that we disregard here (D'Amore, Fandiño Pinilla, Iori, 2013).

7.4. Variable two: the presence of the title of the passage T chosen for the test.
Some of the texts given to the students started with the title, sometimes texts providing the first explanations of a mathematical concept, indeed the one recalled in the title. For example one of the texts had the title: Arithmetic mean. The text said:

> Arithmetic mean
> In statistics, the arithmetic mean is the number obtained…

Deleting the words that are multiples of five but leaving the title unchanged:

> Arithmetic mean
> In statistics, the arithmetic ⋯ is the number obtained…

We could naively believe that the first deleted word (mean) can be easily deduced from the title. But instead only very few students choose to replace the space with “mean”. Many students leave the space empty, others write: sum, expression, calculation and other words.
This brings us to reconsider the sense given to the reading of a passage on the part of the student who accomplishes it. Maybe the editing structure of the text T, that is obvious for the teacher but it is not obvious for all students. The title explains the content to the adult but it does not entail any information for some students (especially for the younger ones).

8. Reading and writing mathematics; communicative learning in mathematics

The classic dichotomy reading/writing leads us to recall that, in mathematical school practice more or less widespread all over the world, students read mathematics (little and improperly, sometimes without understanding) but hardly ever they write mathematics apart from texts of problems, definitions, or rules under dictation. Instead it has been repeatedly noticed that the best way to grasp concepts is to write about them explicitly, and mathematics is not an exception. (D’Amore, Maier, 2002). One of the most successful techniques is the

> «(…) TEP [literally: autonomous students textual productions] i.e. texts autonomously written by students with a mathematical subject. They must not coincide with other productions that are not written autonomously (written tests, notes, description of procedures etc.). The productions of the last examples are not autonomous; they undergo constraints more or less explicitly fixed, and are subjected to direct or indirect evaluations. (…) We consider TEP those productions in which the student, in a condition that fosters his *desire* to express himself in a comprehensible way, feels free from linguistic biases and makes use of spontaneous expressions.
> Therefore examples of TEP are:
> - commented protocols of problem solving (like the ones described above)
> - an account as much as possible spontaneous of mathematical researches (attempts, steps of a procedure, measurements, results, …)
> - detailed descriptions and explanations of concepts or mathematical algorithms
> - texts introduced within a specific situation that requires to communicate mathematical facts and relationships in written form
> - texts that define mathematical concepts, that formulate hypotheses, argumentations, or proofs related to a mathematical theorem or in any case in a mathematical situation
> - (…)» (D’Amore, Maier, 2002).

Amongst the didactical functions of TEP the following are interesting for our research:

> «(…) The production of TEP encourages the student to analyse and reflect on mathematical concepts, relationships, operations, procedures, researches and problem solving processes in which he is involved. In this manner a student can reach a greater awareness and a deeper mathematical understanding of such entities.
> TEP can improve the students’ competences and the performances in using specific language, since they have time for an attentive and reflexive choice of linguistic meanings. Students sometimes do not have this opportunity during a discussion or an oral test since its dialogical evolution is too fast and too much characterized by the pre-established relationship. TEP encourage an active use of technical terms and symbols (Maier, 1989a, 1989b, 1993; Maier, Schweiger, 1999).
> TEP give the students the opportunity to keep under constant control their understanding of mathematical issues, due to a reasoned and reflexive exchange with the teacher and the classmates.

TEP allow the teacher an effective assessment of the personally constructed knowledge and the comprehension of mathematical ideas, in a more detailed and deep way compared to the possibilities offered by the common written texts, usually accomplished as not commented protocols of problem solving activities. (…)» (D'Amore, Maier, 2002).

On the basis of what highlighted in 7. and in the first part of 8. we want to stress that this kind of learning belongs to what elsewhere we called "communicative learning in mathematics" (Fandiño Pinilla, 2008).

It is not by chance that we decided to recall explicitly and decisively the issue of "writing mathematics" and TEP. Delivering to the student a cloze test relative to a mathematical text demands an incomplete and not explicit communication between the author of the text and the pupil. A communication that the student has to integrate/invent/construct/create again. This implies that the student has to make the communication explicit on the basis of supposed intentions of the author that he interprets (or of the teacher who presented the same mathematical content, without excluding possible readings of other authors on the same topic). The student is completing a partial explicit mathematical communication making it, from his point view, total and complete. In this communicative activity he puts into play his comprehension, his interpretation, his own linguistic experience. It is not like writing a TEP that assigns a theme to deal with and a subject who receives the piece. It is very different. He must accept the communication of another person, an author or the teacher and the topic he puts into play.

Nevertheless it is still a form of communication.

If communicating is *cum-munire* (i.e.: bind or construct with), an interesting form of communication arises between the author of a text (chosen in advance, generally an expert), a student (novice) who acts to "cloze" the text interpreting it, and an adult (the teacher, also an expert) who has to decide if the task has been accomplished correctly. There is an exchange of information between the author, the teacher and the student. The student appeals to comprehension and his personal culture to interpret the lacking information, but he communicates only implicitly with the author because he knows that his work undergoes the close examination of the teacher. Therefore in reality he communicates mainly with his teacher. The teacher in judging the choice of the words that cloze the incomplete text communicates with the student, thereby closing the cycle.

9. Conclusions

The fascinating feature of empiric researches is that they can never be said concluded. The change of sample entails a change of some variables. If we repeat the tests we can find results that improve the previous ones and so on and so forth.

For example what happens if the text T the students have to read has been previously discussed in the classroom? What happens if we assign the reading of the text as homework, warning the students that its comprehension will be assessed the following day? And if instead of analysing a text written by an adult the students analysed the text of a classmate (D'Amore et al., 1995)?

The readability tests, unchanged since 1995, acquire in our experience a new vitality, but also the burden of the modern critic awareness that such formulae cannot be considered as an absolute but as a relative (it would be sufficient changing the values of the multiplicative factors). We therefore hope there will be space for future researches in this field.

The choices we made lead to think that our formula is still a classical comprehension formula (not only a readability formula), following the same direction of the first researches carried out between the '50s and the '90s, but with completely new lines of investigation.

But we are aware that the teacher can use our formula:

a) to evaluate as objectively as possible the difficulties of mathematical text comprehension. For example by administrating the test to the students of her/his class to verify the average of success;

b) once the teacher has verified that the text is comprehensible for most of her/his students of her/his class, he/she can find a way to increase the comprehension of the students whose comprehension index is below average.
In this perspective we have at our disposal objective indexes, not only intuitions. When assessment is as much as possible objective, it is easier to help those who are facing difficulties. The teacher has only to understand the causes that have led to failure.

Eventually, although it was not in the authors' objectives, this research has highlighted a didactical gap, a disruption in the teaching-learning praxis. It is necessary to expose students to the individual reading of mathematical passages, avoiding the teachers' strong habit to interpret them. The student needs activities of this kind, he needs to make his own mistakes, correct himself, and reflect on written texts also in view of the tests that he will have to inevitably face by himself.

Therefore we hope that we have given a contribution to a professional reflection on a specific feature of mathematical learning, at whatever school level.

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